# Spectra of random Hermitian matrices with a small-rank external source: supercritical and subcritical regimes

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**Abstract**

Random Hermitian matrices with a source term arise, for instance, in the study of non-intersecting Brownian walkers [1, 20] and sample covariance matrices [4]. We consider the case when the $n \times n$ external source matrix has two distinct real eigenvalues: $a$ with multiplicity $r$ and zero with multiplicity $n-r$. The source is small in the sense that $r$ is finite or $r = \mathcal{O}(n^\gamma)$, for $0 < \gamma < 1$. For a Gaussian potential, Péché [29] showed that for $|a|$ sufficiently small (the subcritical regime) the external source has no leading-order effect on the eigenvalues, while for $|a|$ sufficiently large (the supercritical regime) $r$ eigenvalues exit the bulk of the spectrum and behave as the eigenvalues of $r \times r$ Gaussian unitary ensemble (GUE). We establish the universality of these results for a general class of analytic potentials in the supercritical and subcritical regimes.

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# 1 Introduction

The physical motivation behind studying Hermitian random matrix ensembles is as a model of the Hamiltonian for complex systems, where the eigenvalues of the random Hermitian matrix represent the energy levels of a system without time-reversal invariance [31].

Let $\mathbf{A}$ be a fixed Hermitian matrix. We consider the set of all $n \times n$ Hermitian matrices $\mathbf{M}$ endowed with the probability measure

$$\mu_n(d\mathbf{M}) = \frac{1}{Z_n} e^{-n\mathrm{Tr}(V(\mathbf{M}) - \mathbf{AM})} d\mathbf{M}; \quad Z_n := \int e^{-n\mathrm{Tr}(V(\mathbf{M}) - \mathbf{AM})} d\mathbf{M}, \tag{1-1}$$

where $d\mathbf{M}$ is the entry-wise Lebesgue measure and the integration is over all Hermitian matrices.

When $\mathbf{A} = \mathbf{0}$ (no external source) and $V(\mathbf{M}) = \mathbf{M}^2/2$, (1-1) describes the Gaussian Unitary Ensemble, or GUE. For $\mathbf{A} \neq \mathbf{0}$ and $V(\mathbf{M}) = \mathbf{M}^2/2$, this measure arises in the study of Hamiltonians that can be written as the sum of a random matrix and a deterministic source matrix [16]. We are specifically interested in small-rank sources of the form

$$\mathbf{A} = \mathrm{diag}(\underbrace{a, \dots, a}_{r}, \underbrace{0, \dots, 0}_{n-r}) \tag{1-2}$$

assuming that either $r = \mathcal{O}(n^\gamma)$, $0 < \gamma < 1$ or $r$ is finite (in which case we define $\gamma := 0$). The ratio of $r$ to $n$, which is asymptotically small, will be denoted as

$$\kappa := \frac{r}{n}. \tag{1-3}$$

Péché [29] studied the limiting distribution of the largest eigenvalue in the Gaussian case ($V(\mathbf{M}) = \mathbf{M}^2/2$) under these assumptions and found three distinct behaviors. In the supercritical case, $r$ eigenvalues are expected to exit the bulk and are found to distribute as the eigenvalues of an $r \times r$ GUE matrix. For the subcritical case, the largest eigenvalue is expected to lie at the right band endpoint and behave as the largest eigenvalue of an $n \times n$ GUE matrix. In the critical case, when the outliers lie at the band endpoint, the distribution for the largest eigenvalue is an extension of the standard GUE Tracy-Widom function [30] which arises when $r = 0$ (see also [1, 4, 6]).

One of the primary goals of random matrix theory is to determine universality classes of matrix ensembles, that is, find different probability measures on the space of matrices for which the spectral properties are the same in the large-$n$ limit. With this goal in mind we consider more general functions $V(\mathbf{M})$, with our specific assumptions listed in Section 1.3. Basically, we assume $V(\mathbf{M})$ is a generic single-gap analytic potential with sufficient growth at infinity. We show that Péché's results [29] hold for these general potentials in the supercritical and subcritical cases. The universality of the critical case will be considered elsewhere [8].

## 1.1 Definition of the supercritical, subcritical, and critical regimes

Let $g(z)$ be the $g$-function associated with the orthogonal polynomials with potential $V(z)$ (see, for instance, [21] or [23]). It may be written as

$$g(z) := \int_{\mathbb{R}} \log(z-s)\rho_{\min}(s)ds, \tag{1-4}$$

where $\rho_{\min}$ is the unique probability measure minimizing the functional

$$\mathcal{F}[\rho] := \int_{\mathbb{R}} V(s)\rho(s)ds - \int_{\mathbb{R}}\int_{\mathbb{R}} \rho(s)\rho(s')\log|s-s'|dsds'. \tag{1-5}$$

We will assume this equilibrium measure $\rho_{\min}$ is supported on a single band $[\alpha,\beta]$ (see Assumption 1.1 (v) in Section 1.3). Define

$$P_1(z) \;:=\; -V(z) + 2g(z) + l_1, \tag{1-6}$$

$$P_2(z) \;:=\; -V(z) + az + g(z) + l_2, \tag{1-7}$$

$$P_3(z) \;:=\; -P_1(z) + P_2(z) = az - g(z) - l_1 + l_2. \tag{1-8}$$

Here $g(z)$ and $l_1$ are uniquely determined by the conditions that $P_1(z)_\pm$ is purely imaginary on the support of the equilibrium measure and has negative real part on its complement in $\mathbb{R}$ (here the subscripts $_\pm$ denote the boundary values from above/below the real axis). How $l_2$ is chosen will be described at the end of this section.

It is also known that $\Re g(z)$ is a **continuous** function on $\mathbb{R}$ and **harmonic** on the complement of the support of $\rho_{\min}$ (up to a sign it is also known as the *logarithmic potential* in potential theory).

**Definition 1.1.** *Define $a_c$ to be the (unique) value of $a$ so that $P_2'(\beta) = 0$.*

The uniqueness is promptly seen because $P_2'(\beta) = -V'(\beta) + a + g'(\beta)$; in fact the effective potential $P_1$ is known [22] to satisfy

$$P_1'(z) = \mathcal{O}(z-\beta)^{\frac{1}{2}} \tag{1-9}$$

and in particular $P_1'(\beta) = 0$ and hence $P_2'(\beta) = a - g'(\beta) = a - \frac{1}{2}V'(\beta)$. Thus the critical value of $a$ is given by

$$a_c = g'(\beta) = \frac{1}{2}V'(\beta)\,. \tag{1-10}$$

We first have

**Lemma 1.1.** *The critical value $a_c = g'(\beta)$ is positive. Moreover $g'(\alpha) < 0$.*

*Proof.* From (1-4) we see that $g'(z) = \int_\alpha^\beta \frac{1}{z-s}\rho_{\min}(s)\mathrm{d}s$ is *positive* for $z > \beta$. It is also known that the density $\rho_{\min}$ vanishes like a square root at the endpoints $\alpha, \beta$ and hence the integral representation of $g'(\beta)$ is convergent and immediately shows it to be positive. Similarly $g'(\alpha)$ is *negative.* Note that this proof does not require the support to consists of a single band as long as we understand $\beta = \sup \operatorname{supp} \rho_{\min}$ and $\alpha = \inf \operatorname{supp} \rho_{\min}$. $\square$

Lemma 1.1 implies that there is no loss of generality in studying only the case $a > 0$ since there is always a positive critical $a_c$ (and a negative one); the negative case ($a < 0$) is equivalent to the positive case by replacing $a \mapsto -a$, $V(z) \mapsto V(-z)$.

**Lemma 1.2.** *The critical point structure of $\Re P_3(z)$ is:*

- *For $a > a_c$, $\Re P_3(z)$ is strictly increasing on $\mathbb{R} \setminus [\alpha, \beta]$;*
- *For $a = a_c$, $\Re P_3(z)$ is strictly increasing on $\mathbb{R} \setminus [\alpha, \beta]$ and $\Re P_3'(\beta) = 0$;*
- *For $0 < a < a_c$, $\Re P_3(z)$ has unique local minimum on $\mathbb{R} \setminus (\alpha, \beta)$. This minimum occurs at a point $b^\star \in (\beta, \infty)$.*

*Proof.* From the representation (1-4) of $g$ one sees immediately that

$$g''(z) = -\int_{\mathbb{R}} \frac{1}{(z-s)^2}\rho_{\min}(s)\mathrm{d}s \tag{1-11}$$

which shows clearly that for $z \in \mathbb{R} \setminus \operatorname{supp} \rho_{\min}$ the real part of $g$ is concave downward. Thus $\Re g'(z)$ is decreasing in $\mathbb{R} \setminus \operatorname{supp} \rho_{\min}$; moreover, from

$$g'(z) = \int_{\mathbb{R}} \frac{1}{(z-s)}\rho_{\min}(s)\mathrm{d}s \tag{1-12}$$

we see that $\Re g'$ is negative for $z < \inf \operatorname{supp} \rho_{\min} = \alpha$ and positive for $z > \sup \operatorname{supp} \rho_{\min} = \beta$.

From the definition we see that $P_3'(z) = a - g'(z)$ and hence we infer:

- For $a > a_c = g'(\beta) > 0$, $\Re(P_3'(z)) = a - g'(z) > g'(\beta) - g'(z)$ is positive on $[\beta, \infty)$ therefore $\Re P_3$ is strictly increasing. On the other hand $a - g'(z)$ is clearly positive on $(-\infty, \alpha]$ because $a > 0$ and $-g' > 0$ from (1-12);
- For $a = a_c$, $\Re P_3'(\beta) = 0$ and $\Re P_3'(z)$ is a monotonically increasing positive function on $(\beta, \infty)$, and a monotonically increasing positive function on $(-\infty, \alpha]$. Therefore there is a single critical point of $\Re P_3(z)$ at $z = \beta$. As $\Re P_3''(z) = -\Re g''(z) > 0$ this must be a minimum;
- For $0 < a < a_c$, $\Re P_3'(\beta) < 0$ and $P_3'(z) \to a > 0$ for $z \to \infty$; moreover $\Re P_3'(z)$ is a monotonically increasing function on $[\beta, \infty)$, and a (monotonically increasing) positive function on $(-\infty, \alpha]$. Since $P_3'(\beta) < 0$ there must be a unique point $b^\star > \beta$ where $P_3'(b^\star) = 0$. As $\Re P_3''(z) = -\Re g''(z) > 0$ this must be the local minimum (or, equivalently, the global minimum on $(\beta, \infty)$).

□

We can now define four regimes: supercritical, subcritical, critical, and jumping outliers. We define the subcritical and critical regimes first.

**Definition 1.2.** *The matrix model specified by* (1-1) *is in the* **subcritical regime** *if $a < a_c$ and $P_2(x) < P_3(b^\star)$ for all $x \geq b^\star$.*

**Definition 1.3.** *The matrix model specified by* (1-1) *is in the* **critical regime** *if $a = a_c$ and $P_2(x) < P_2(\beta)$ for all $x > \beta$.*

Now the supercritical regime can be efficiently defined as the remaining cases, except the small—codimension one—cases that we distinguish as the "jumping outlier regime."

**Definition 1.4.** *The model is in the* **supercritical regime** *if $P_2$ has a unique point of global maximum on $\{x > \max\{\beta, b^\star\}\}$ at a point $x = a^\star \in \mathbb{R}$ and any of the three conditions below is satisfied:*

- *$a > a_c$.*
- *$a = a_c$ and $P_2(\beta) < P_2(x)$ for some $x > \beta$.*
- *$0 < a < a_c$ and $P_3(b^\star) < P_2(x)$ for some $x > b^\star$.*

*Note that $a^\star$ is always greater than $\beta$ and $b^\star$.*

*If the global maximum of $P_2$ on $(\max\{\beta, b^\star\}, \infty)$ is attained at several distinct points then we will say that we are in the* **jumping outlier regime** *that also includes the following remaining case.*

- *$0 < a < a_c$ and $P_2(x) = P_3(b^\star)$ for some $x > b^\star$. (The case $x = b^\star$ cannot occur for* **regular** *$V$.)*

In the present paper we consider the supercritical and subcritical regimes. The critical and jumping outlier regimes will be considered elsewhere [8].

The definition of the supercritical regime is complicated and the reader may wonder whether the above definitions ever hold in actual examples. It is however not difficult to engineer a situation where they do occur, explained in the following example

**Example 1.1** (Second and third bullet in Definition 1.4)**.** *Consider a potential $V$ such that a new spectral band (i.e. interval of support of $\rho_{min}$) is about to emerge. Then $P_1$ has a local maximum $-E$ at $x_0 > \beta$ outside of the main band which is slightly negative but small in absolute value. It is simple to arrange examples where $E$ is arbitrarily small. Since $a = a_c$ we have $\Re P_3'(\beta) = 0$ (see* (1-10)*) and since $\Re P_3$ is concave upwards, it must be increasing for $x > \beta$. On the other hand $P_2 = P_1 + P_3$ and thus*

$$\Re\left(P_2(x_0) - P_2(\beta)\right) = -E + \Re(P_3(x_0) - P_3(\beta)). \tag{1-13}$$

*Since $E$ can be chosen arbitrarily small, and since $\Re(P_3(x_0) - P_3(\beta)) > 0$, we see that necessarily we can have the situation described in the second bullet. By a continuity argument on $a < a_c$, this also provides an example for the third bullet since $P_3(b^\star) < P_3(\beta) \approx P_2(\beta) < P_2(x_0)$.*

Though we do not consider in this paper, one can also create an example in the jumping outlier regime.

We show in Proposition 1.1 that if $V$ is convex, then we are either in the super or subcritical depending on $a > a_c$ or $0 \le a < a_c$, respectively; in particular, in this case, the situation described in the second and third bullet points of Definition 1.4 cannot occur.

**Proposition 1.1.** *Suppose that $V(z)$ satisfies Assumptions 1.1 and in addition it is convex ($V'' > 0$). Then*

- **(i)** *for $a > a_c$ the model is supercritical and the maximum of $P_2$ at $a^\star$ is nondegenerate;*
- **(ii)** *for $0 < a < a_c$ the model is subcritical;*
- **(iii)** *there is no jumping outlier regime.*

*Proof.* It is known that the convexity of $V$ is a sufficient condition for the support of the equilibrium measure to be a single band $[\alpha, \beta]$. Moreover from Lemma 1.1 we see $V'(\beta) = 2g'(\beta) > 0 > V'(\alpha) = 2g'(\alpha)$. Note that $P_j$ are all real-valued in $[\beta, \infty)$ (the cut of the logarithm runs in $(-\infty, \beta]$).
**(i)** We then observe that

$$P_2'' = -V'' + g'' < 0 \tag{1-14}$$

since both $-V$ (by assumption) and $g$ (by (1-11)) are concave downward. $P_2$ may have at most a single global (nondegenerate) maximum in $[\beta, \infty)$ because $P_2'(\beta) = a - \frac{1}{2}V'(\beta) = a - a_c > 0$. This also proves (**iii**).
**(ii)** If $0 < a < a_c$, $P_2$ strictly decreases on $[\beta, \infty)$ because $P_2'(\beta) = a - \frac{1}{2}V'(\beta) = a - a_c < 0$.; Also we have $P_2(b^\star) = P_1(b^\star) + P_3(b^\star) < P_3(b^\star)$. Therefore $P_2(x) < P_3(b^\star)$ for all $x \ge b^\star$ as in Definition 1.2. ☐

It should be noted here that, contrary to the work done for $V = z^2/2$, the position of $a$ relative to $a_c$ is not sufficient (for general $V$) to define the critical and subcritical regimes. If, however, $V$ is convex (for example an even monomial with positive coefficient) then by Proposition 1.1 the position of $a$ relative to $a_c$ determines the supercritical/subcritical regime as in [14, 3, 15, 1, 29]. The secondary conditions in Definition 1.4 are dealing with whether the Lagrange multiplier $\ell_2$ in the effective potentials $P_j$ can be chosen such that the off diagonal entries of the jump matrices for the deformed Riemann-Hilbert problems that we will construct in Sections 2, 3, and 4, decay to zero at an exponential rate. The problem of finding necessary and sufficient conditions on $V(x)$ and $a$ for the matrix model to be in the supercritical/subcritical regime is quite difficult, as much as it is difficult to find necessary and sufficient conditions for $V(x)$ to be a single-band potential. We now specify the constant $l_2$:

**Definition 1.5.** *The constant $l_2$ in Definition 1-7 will be chosen as follows:*

- *In the supercritical case, the constant $l_2$ is chosen so that the unique global maximum of $P_2(z)$ on $(\beta, \infty)$ is zero (i.e. $\Re P_2(a^\star) = 0$).*
- *In the subcritical case, the constant*

  $$l_3 := -l_1 + l_2 \tag{1-15}$$

  *is chosen so that $P_3(b^\star) = 0$.*

## 1.2 The kernel and its connection to multiple orthogonal polynomials

Let $\rho_m(\lambda_1, \ldots, \lambda_m)$ be the probability density that the $n \times n$ matrix $\mathbf{M}$ chosen using (1-1) has eigenvalues $\{\lambda_1, \ldots, \lambda_m\}$ (here $m \le n$). Then, the $m$-point correlation function is $R_m(\lambda_1, \ldots, \lambda_m) := \frac{n!}{(n-m)!}\rho_m(\lambda_1, \ldots, \lambda_m)$. Brézin and Hikami [16, 17, 18, 19] showed that, in the Gaussian case, the $m$-point correlation functions can all be expressed in terms of a single kernel $K(x, y)$:

$$R_m(\lambda_1, \ldots, \lambda_m) = \det(K(\lambda_i, \lambda_j))_{i,j=1,\ldots,n}. \tag{1-16}$$

Zinn-Justin [32, 33] extended this result to the case of more general $V(\mathbf{M})$. Bleher and Kuijlaars [13] rewrote the kernel in terms of *multiple orthogonal polynomials*, a significant result because it allows one to analyze the asymptotic behavior of these polynomials via the associated Riemann-Hilbert problem.

This approach was followed by Aptekarev, Bleher, and Kuijlaars [14, 3, 15] in the Gaussian case when the matrix $\mathbf{A}$ has two eigenvalues $\pm a$, each of multiplicity $n/2$. When $a$ is sufficiently large the eigenvalues of $\mathbf{M}$ accumulate on two disjoint intervals (the supercritical case). As $a$ decreases the two bands collide (the critical case). Below this critical value of $a$, the eigenvalues accumulate on a single interval (the subcritical case). Related behavior also appears in the theory of nonintersecting one-dimensional Brownian motions; see, for instance, Adler, Orantin, and van Moerbeke [2] for the critical case.

In general, the existence and number of bands on which eigenvalues accumulate for large-rank sources for general $V(\mathbf{M})$ is a complicated problem. For more on this question see McLaughlin [28] in which the quartic case $V(\mathbf{M}) = \mathbf{M}^4/4$ is worked out. Bleher, Delvaux, and Kuijlaars [12] have studied the external source problem with two eigenvalues of equal multiplicity and where $V(\mathbf{M})$ is a sum of even-degree monomials with positive coefficients. The external source with a finite number of different eigenvalues with various multiplicity for supercritical case has been considered in [25].

The starting point of our analysis is the Riemann-Hilbert problem associated to the multiple orthogonal polynomials. Suppose $\mathbf{Y}(z)$ is a $3 \times 3$ matrix-valued function of the complex variable $z$ satisfying

$$\begin{cases} \mathbf{Y}(z) \text{ is analytic for } z \notin \mathbb{R}, \\ \mathbf{Y}_+(x) = \mathbf{Y}_-(x) \begin{pmatrix} 1 & e^{-nV(x)} & e^{-n(V(x)-ax)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ for } x \in \mathbb{R}, \\ \mathbf{Y}(z) = \left(\mathbf{I} + O\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 & 0 \\ 0 & z^{-(n-r)} & 0 \\ 0 & 0 & z^{-r} \end{pmatrix} \text{ as } z \to \infty. \end{cases} \tag{1-17}$$

Here $\mathbf{Y}_\pm(x) := \lim_{\varepsilon \to 0} \mathbf{Y}(x \pm i\varepsilon)$ denote the non-tangential limits of $\mathbf{Y}(z)$ as $z$ approaches the real axis from the upper and lower half-planes. Whenever posing a Riemann-Hilbert problem we assume (unless otherwise stated) that the solution has continuous boundary values along the jump contour when approached from either side. Under our assumption (iv) in Section 1.3, the unique solution $\mathbf{Y}(z)$ can be written explicitly in terms of multiple orthogonal polynomials of the second kind (see [14], Section 2). In the case of two distinct eigenvalues $a$ and 0, the kernel $K_n(x, y)$ may be written in terms of the function $\mathbf{Y}(z)$ as

$$K_n(x, y) = \frac{e^{-\frac{1}{2}n(V(x)+V(y))}}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 & e^{nay} \end{pmatrix} \mathbf{Y}(y)^{-1}\mathbf{Y}(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{1-18}$$

In the technical analysis of this Riemann-Hilbert problem we use and improve certain ideas introduced by Bertola and Lee [9] to study the first finitely many eigenvalues in the birth of a new spectral band for the random Hermitian matrix model without source.

We note here that Baik [7] has recently expressed the kernel $K_n(x, y)$ in terms of the standard (not multiple) orthogonal polynomials. This offers an alternative method for approaching the problem we consider here. Based on this approach, Baik and Dong [5] have obtained the universality result similar to ours, for the case of finite $r$ but possibly for non-degenerate eigenvalues, i.e. $\mathbf{A} = \mathrm{diag}(a_1, a_2, \cdots, a_r, 0, \cdots, 0)$. Since the rank of the matrices involved in this alternative formulation grow with $r$, analyzing the Riemann-Hilbert problem (1-17) seems more feasible if $r$ is allowed to grow sublinearly with $n$.

## 1.3 Assumptions on A and V(z) and results

First we gather the assumptions we will make in the rest of the paper.

**Assumption 1.1.** *We make the following requirements*

*(i)* $a > 0$.

*(ii)* $\mathbf{A}$ *is a* small-rank *external source of the form* (1-2) *with either* $r$ *a fixed positive integer or* $r = \mathcal{O}(n^\gamma)$, *with* $0 \le \gamma < 1$. *When* $r$ *is fixed we say* $\gamma = 0$.

*(iii)* $V(z)$ *is real-analytic.*

*(iv)* $\displaystyle\lim_{|z|\to\infty} \frac{V(z)}{\log(1+z^2)} = \infty$ *and* $\displaystyle\lim_{|z|\to\infty} \frac{V(z) - az}{\log(1+z^2)} = \infty$.

*(v)* $V(z)$ *is a* single-band *potential (for example it can be convex).*

*(vi) The density of the equilibrium measure of* $V(z)$ *has square root decay at its two endpoints (i.e. it is* regular *in the sense of [23]).*

*(vii) For the supercritical regime,* $P_2(z)$ *behaves quadratically near* $a^\star$. *Specifically,*

$$-V'(z) + a + g'(z) = -c(z - a^\star) + O((z - a^\star)^2) \text{ as } z \to a^\star \tag{1-19}$$

*for some constant* $c > 0$.

Assumption (i) is for convenience, as the case when $a < 0$ is equivalent by sending $a \to -a$ and $V(z) \to V(-z)$. Regarding assumption (ii), in the general case when $\mathbf{A}$ has $m > 2$ distinct eigenvalues the kernel can be written in terms of multiple orthogonal polynomials associated to an $(m+1) \times (m+1)$ Riemann-Hilbert problem, which is beyond the scope of this paper. Assumption (iii) allows us to use the nonlinear steepest-descent method for Riemann-Hilbert problems, while (iv) guarantees the existence of the multiple orthogonal polynomials needed to ensure the Riemann-Hilbert problem has a solution.

Assumption (v) avoids the necessity of using Riemann-theta functions for the solution of the outer model Riemann-Hilbert problem. We expect similar results to hold generically in the multi-band case.

Both (vi) and (vii) are genericity assumptions. Assumption (vi) allows us to use Airy parametrices near the band endpoints. Assumption (vii) produces Hermite (or Gaussian) behavior of the outlying zeros. Our results are computations of the large $n$ behavior of the kernel function (1-18). We explicitly compute the kernel in a neighborhood of $a^\star$ for the supercritical regime and in a neighborhood of $b^\star$ for the subcritical regime. In the remaining portions of the complex plane, our result is that the kernel function converges to the kernel for the classical orthogonal polynomial problem with respect to $V(x)$. In particular, our results include that, away from the $a^\star$ and $b^\star$, the standard universality classes apply (i.e. a sine kernel in the bulk of the spectrum, and Airy kernels at the edges).

**Theorem 1.1.** *Suppose $V(z)$ and $a$ satisfy assumptions (i)–(vii) and definition 1.4 of the supercritical regime. Let $\zeta_x$ and $\zeta_y$ be the local coordinates corresponding to $x$ and $y$ near $a^\star$ as defined in* (2-42)*. Uniformly for $\zeta_x, \zeta_y$ in compact sets we have the following asymptotics for $r = Cn^\gamma$ for some $C > 0$, $0 \le \gamma < 1$.*

$$K_n(x(\zeta_x), y(\zeta_y)) = e^{-\frac{n}{2}P_3(x)+\frac{n}{2}P_3(y)} \frac{\sqrt{f''(0)}}{k_{r-1}^{(r)}} \kappa^{-1/2} \left(K_r^{GUE}(\zeta_x,\zeta_y) + \mathcal{O}(n^{-(1-\gamma)/2})\right), \tag{1-20}$$

*where $P_3(x)$ is given by* (1-8)*, $f(z;\kappa)$ is defined in* (2-56)*, $k_{r-1}^{(r)}$ is defined by* (2-62)*, $\kappa = r/n$, and*

$$K_r^{GUE}(\zeta_x,\zeta_y) := \frac{H_r^{(r)}(\zeta_x-\zeta_0)H_{r-1}^{(r)}(\zeta_y-\zeta_0) - H_{r-1}^{(r)}(\zeta_x-\zeta_0)H_r^{(r)}(\zeta_y-\zeta_0)}{\zeta_x-\zeta_y} e^{-\frac{r}{4}(\zeta_x-\zeta_0)^2-\frac{r}{4}(\zeta_y-\zeta_0)^2} \tag{1-21}$$

*is the kernel for $r$ eigenvalues of the Gaussian Unitary Ensemble of scale $r$ centered at $\zeta_0$, which is defined by the change of variables* (2-58)*. Here $H_i^{(r)}(\zeta)$ are the rescaled monic Hermite polynomials satisfying the orthogonality condition* (2-62)*.*

The presence of $\exp(-nP_3(x)/2)$ in (1-20) does not affect spectral properties of the kernel (because it amounts to a conjugation of the kernel by a diagonal operator) and therefore the implication is that asymptotically the eigenvalues near $a^\star$ are equivalent to those of a scaled $r \times r$ GUE problem; if $V(x)$ is a quadratic potential this agrees with the results of [29].

**Remark 1.1.** *If the critical point of $P_2$ at $a^\star$ is more degenerate, $P_2(z) = \mathcal{O}((z-a^\star)^{2k})$, then one may follow similar steps as in [9] and [11] and conclude that the relevant statistics of the outliers are determined by the kernel of a unitary ensemble with potential given by a polynomial of degree $2k$ instead of a Gaussian, namely, obtained from a deformation of the Freud orthogonal polynomials.*

Theorem 1.1 shows that, as expected, the equilibrium measure has no mass near $a^\star$. We have the mean density of states

$$\rho_n(x(\zeta_x)) = \lim_{\zeta_y\to\zeta_x} \frac{1}{n} K_n(x(\zeta_x), y(\zeta_y)) = \frac{\sqrt{f''(0)}}{k_{r-1}^{(r)}} \kappa^{1/2} \left(\rho_r^{(r)}(\zeta_x) + \mathcal{O}(n^{-(1-\gamma)/2} r^{-1})\right) \tag{1-22}$$

where $\rho_r^{(r)}(\zeta)$ is the mean density of eigenvalues for the $r \times r$ GUE ensemble. If $r$ is fixed, this quantity is $\mathcal{O}(n^{-1/2})$ for large $n$, and if $r = n^\gamma$, using that $\rho_r^{(r)}(\zeta_x) \to \frac{1}{2\pi}\sqrt{4-\zeta_x^2}$ as $r \to \infty$, we find

$$\lim_{n\to\infty} \rho_n(x(\zeta_x)) = \frac{\sqrt{f''(0)}}{2\pi k_{r-1}^{(r)}} \kappa^{1/2} \sqrt{4-\zeta_x^2}\,. \tag{1-23}$$

In either case, our conclusion is that for large $n$ the mean density of eigenvalues is asymptotically small (of order $\kappa^{1/2}$) in the neighborhood of $a^\star$ chosen in the theorem. See Chapters 5 and 6 of [21], Theorem 1.1 of [14], and Theorem 8.1 of [28] for similar results regarding the derivation of the asymptotic mean distribution of eigenvalues from the kernel.

**Theorem 1.2.** *Suppose the pair $(V(z), a)$ satisfies Definition 1.2 of the subcritical regime. There exists a closed disk of fixed radius centered at $b^\star$ such that, for $x$ and $y$ in this disk, for large $n$, and for $r = Cn^\gamma$ for some $C > 0$ and $0 \le \gamma < 1$, there is a $c > 0$ such that*

$$K_n(x, y) = \mathcal{O}(n^{-(1-\gamma)/2} e^{-cn}). \tag{1-24}$$

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# 2 The supercritical regime

## 2.1 Modified equilibrium problem

In this section we will use a positive integer $K$; the general statements are valid for any $K$ but we will choose (for future use)

$$K \ge \max\left\{\frac{3\gamma - 1}{1-\gamma}, 0\right\} \tag{2-1}$$

where, we recall, $\gamma$ is the exponent of growth of $r = Cn^\gamma$ for some $C$ and $0 \le \gamma < 1$.

We will need to build a *perturbation* of the equilibrium problem that leads to the definition of $g(z)$; we will denote by $\mathfrak{g}$ the resulting $g$–function of this perturbation scheme. The construction, rather involved, will be broken down in steps.

The unperturbed equilibrium measure is supported on the single interval $[\alpha, \beta]$ (by assumption) with external field $V(z)$. Recall that $a^\star$ is lying *outside* of $[\alpha, \beta]$ and we fix a compact interval $J$ containing $[\alpha, \beta]$ in its interior and $a^\star \notin J$.

**Proposition 2.1.** *For any $K \in \mathbb{N}$ there is a neighborhood of the origin in $(\kappa, \vec{\delta}) \in \mathbb{C}^{1+K}$ such that the equilibrium measure $\widetilde{\sigma}(x)\mathrm{d}x$ of total mass $1-\kappa$ for the external field*

$$\widetilde{V}(z) := V(z) - \delta V(z), \quad \delta V(z) := \kappa \ln(z - a^\star) + \kappa \sum_{j=1}^{K} \frac{\delta_j}{2(z - a^\star)^j} \tag{2-2}$$

*is supported on a single interval $[\alpha(\kappa, \vec{\delta}), \beta(\kappa, \vec{\delta})]$ still contained in the interior of $J$: the endpoints $\alpha(\kappa, \vec{\delta}), \beta(\kappa, \vec{\delta})$ are analytic functions of the specified variables.*

*Furthermore the (normalized) $g$–function of this problem*

$$\mathfrak{g}(z) := \int \ln(z - w) \frac{\widetilde{\sigma}(w)}{1 - \kappa} \mathrm{d}w \tag{2-3}$$

*converges uniformly over closed subsets not containing $[\alpha, \beta]$ to the unperturbed $g$–function.*

**Remark 2.1.** *In this proposition we treat the deformation parameters $\kappa, \vec{\delta}$ as independent from each other; later on, in Proposition 2.2, they will be uniquely determined in terms of the sole parameter $\kappa$.*

*Proof.* It is well known (see, for example, [22]) that

- $\mathfrak{g}(z)$ is analytic for $z \notin (-\infty, \widetilde{\beta}]$, where $\widetilde{\beta} = \sup\ \mathrm{supp}(\widetilde{\sigma})$;
- $\mathfrak{g}(z)$ has continuous boundary values and satisfies

$$\begin{aligned} &\mathfrak{g}_+(z) - \mathfrak{g}_-(z) = 2\pi i, \ z \in (-\infty, \widetilde{\alpha}), \\ &(1-\kappa)\left(\mathfrak{g}_+(z;\kappa) + \mathfrak{g}_-(z;\kappa)\right) = V(z) - \kappa \log(z - a^\star) - \sum_{j=1}^{2k} \frac{\delta_j}{2(z - a^\star)^j} - \ell_1, \ z \in (\widetilde{\alpha}, \widetilde{\beta}) \end{aligned} \tag{2-4}$$

  where $\widetilde{\alpha} = \inf\ \mathrm{supp}(\widetilde{\sigma})$ with the real axis oriented left to right;
- $\mathfrak{g}(z;\kappa) = \log(z) + \mathcal{O}\left(\frac{1}{z}\right)$ as $z \to \infty$.

Vice versa, the $g$–function may be characterized by the scalar Riemann–Hilbert problem (2-4) with the additional requirement that $\Im \mathfrak{g}_+$ is a nondecreasing function.

To show the analytic dependence of $\mathfrak{g}$ on $\kappa, \vec{\delta}$ we proceed as follows: define the function

$$R(z) := ((z - \widetilde{\alpha})(z - \widetilde{\beta}))^{1/2}, \tag{2-5}$$

where the principal branch of the square root is chosen so $R(z;\kappa) = z + \mathcal{O}(1)$ as $z \to \infty$. Taking the derivative of (2-4) with respect to $z$ and using the Plemelj formula gives

$$\mathfrak{g}'(z) = \frac{R(z)}{2\pi i(1-\kappa)} \int_{\widetilde{\alpha}}^{\widetilde{\beta}} \frac{V'(s) - \kappa/(s - a^\star) + \sum_{j=1}^{2k} j\delta_j/(2(s - a^\star)^{j+1})}{(s - z)R_+(s)} ds \tag{2-6}$$

where $R_+(s)$ refers to the limit in $s$ from the upper half-plane. The large-$z$ expansion of (2-6) along with the condition

$$\mathfrak{g}'(z;\kappa) = \frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right) \tag{2-7}$$

gives two conditions on $\alpha(\kappa), \beta(\kappa)$:

$$\int_{\widetilde{\alpha}}^{\widetilde{\beta}} \frac{V'(s) - \kappa/(s-a^\star) + \sum_{j=1}^{2k} j\delta_j/(2(s-a^\star)^{j+1})}{R_+(s)} ds = 0 \tag{2-8}$$

and

$$\frac{1}{2\pi i}\int_{\widetilde{\alpha}}^{\widetilde{\beta}} \frac{V'(s) - \kappa/(s-a^\star) + \sum_{j=1}^{2k} j\delta_j/(2(s-a^\star)^{j+1})}{R_+(s)} s\, ds = 1-\kappa\,. \tag{2-9}$$

These two equations uniquely determine $\widetilde{\alpha}, \widetilde{\beta}$ as analytic functions of the parameters by the implicit function theorem.

The inequality $\Im \mathfrak{g}'_+ > 0$ remains valid, using a continuity argument, for suitably small values of $\kappa, \vec{\delta}$ because it is valid (with the strict inequality) for the unperturbed $g$–function (by our initial assumption).

Therefore the expression (2-6) yields a *bona fide* $g$–function for the modified external field $\widetilde{V}$ in a neighborhood of $(\kappa, \vec{\delta}) = (0, \vec{0})$.

The expression for $\mathfrak{g}$ may be obtained by integration; specifically

$$\mathfrak{g}(z) = \int_{\widetilde{\alpha}}^{z} \mathfrak{g}'(s)\mathrm{d}s - \ell_1 \tag{2-10}$$

and $\ell_1$ is also determined by the requirement that $\mathfrak{g}(z) = \ln(z) + \mathcal{O}(z^{-1})$ (without the constant term). Explicitly

$$\ell_1 = \lim_{z\to\infty}\left(\int_{\widetilde{\alpha}}^{z} \mathfrak{g}'(s)\mathrm{d}s - \ln z\right) \tag{2-11}$$

which expression shows that $\ell_1$ is also analytic in the parameters, given the already proven analyticity of $\mathfrak{g}'$.

The statement about the convergence follows easily by noticing that $\mathfrak{g}'$ converges to $g'(z)$ as desired (note that they both have behavior $1/z + \mathcal{O}(z^{-2})$). This is best seen from the integral representations and the already proven analytic dependence on the deformation parameters. □

In parallel with the definition of the functions $P_j$ we shall define

$$\mathcal{P}_1(z;\kappa) := -\widetilde{V}(z) + 2(1-\kappa)\mathfrak{g}(z;\kappa) + \ell_1, \tag{2-12}$$

$$\mathcal{P}_2(z;\kappa) := -\widetilde{V}(z) + az + \delta V + (1-\kappa)\mathfrak{g}(z;\kappa) + l_2, \tag{2-13}$$

$$\mathcal{P}_3(z;\kappa) := -\mathcal{P}_1(z;\kappa) + \mathcal{P}_2(z;\kappa) \tag{2-14}$$

$$\widetilde{V}(z) := V(z) - \delta V(z), \quad \delta V(z) := \kappa \ln(z-a^\star) + \kappa\sum_{j=1}^{K}\frac{\delta_j}{2(z-a^\star)^j} \tag{2-15}$$

where $l_2$ as in Definition 1.5 and is independent of the deformation.

Using deformation and continuity arguments we have that (for a sufficiently small value of the deformation parameters $\kappa, \vec{\delta}$) the real part of $\mathcal{P}_2$ has a global maximum in a neighborhood of $a^\star$. The main tool in the analysis of the supercritical case when $r$ grows shall be the next theorem.

**Theorem 2.1.** *There exists a conformal change of coordinate $\rho = \rho(z; \kappa, \vec{\delta})$ fixing $z = a^\star$ (i.e. $\rho(a^\star; \kappa, \vec{\delta}) \equiv 0$) that depends analytically on the parameters $\kappa, \vec{\delta}$ such that*

$$\mathcal{P}_2(z) := -V(z) + az + (1-\kappa)\mathfrak{g} + l_2 + 2\kappa \ln(z - a^\star) + \sum_{j=1}^{K} \frac{\delta_j}{(z-a^\star)^j} \tag{2-16}$$

*can be written as*

$$\mathcal{P}_2(z; \kappa, \vec{\delta}) = -\frac{1}{2}(\rho - \mathfrak{a})^2 + 2\kappa \ln \rho + \mathfrak{b} + \sum_{j=1}^{K} \frac{\gamma_j}{\rho^j} \tag{2-17}$$

*where the parameters $\mathfrak{a} = \mathfrak{a}(\kappa, \vec{\delta})$, $\mathfrak{b} = \mathfrak{b}(\kappa, \vec{\delta})$ and $\vec{\gamma} = \vec{\gamma}(\kappa; \vec{\delta})$ are analytic functions of the indicated parameters. Furthermore the Jacobian*

$$\frac{\partial \vec{\gamma}}{\partial \vec{\delta}} \tag{2-18}$$

*is nonsingular in a neighborhood of the origin (i.e. for $\kappa$ and $\vec{\delta}$ sufficiently small).*

*Proof.* To simplify the notation we set $a^\star = 0$ (up to a translation this entails no loss of generality). We can write $\mathcal{P}_2$ as

$$\mathcal{P}_2 = -f(z; \kappa, \delta) + 2\kappa \ln(z) + \sum_{j=1}^{K} \frac{\delta_j}{z^j}\,. \tag{2-19}$$

By the definition of $a^\star$ (which is now translated to 0), the function $f(z; \kappa, \vec{\delta})$ has the property that

$$f(z; 0, \vec{0}) = \frac{C}{2} z^2 (1 + \mathcal{O}(z)) \ , C > 0 \tag{2-20}$$

and hence

$$f(z; \kappa, \vec{\delta}) = \frac{C}{2} z^2 (1 + \mathcal{O}(z)) + \mathcal{O}(\kappa, \vec{\delta}). \tag{2-21}$$

Let us fix any (smooth) curve in the parameter space $\kappa(t), \vec{\delta}(t)$ and denote by a $\partial$ its tangent vector; we then must show the identity (2-17) for $t$ near 0. We suppress the notation of the dependence on $\kappa, \vec{\delta}$ for brevity in what follows, with the understanding that $f(z), \mathfrak{a}, \gamma_j, \mathfrak{b}$ all depend on them.

In this part we only sketch the main idea, leaving a full proof for Appendix A. Let $\mathbb{D}(r)$ be the open disk of radius $r > 0$ and let $\Omega_1$ be the Banach **manifold** of **univalent, analytic functions** $\rho : \mathbb{D}(r) \to \mathbb{C}$ which fix the origin $\rho(0) = 0$; this is a closed Banach submanifold of all univalent analytic functions because the evaluation map is continuous. Define now

$$\mathcal{M} := \Omega_1 \times \mathbb{C}^{K+1} = \{\mathbf{p} = (\rho, \mathfrak{a}, \mathfrak{b}, \vec{\gamma}), \quad \rho \in \Omega_1, \quad \mathfrak{a}, \mathfrak{b} \in \mathbb{C}\} \tag{2-22}$$

which is naturally also an *infinite dimensional* Banach manifold. We are going to show that the ordinary differential equation in $\partial$ that derives from (2-17) is integrable on $\mathcal{M}$; taking the implicit differentiation of (2-17) we obtain

$$\begin{aligned}
-\partial f(z) + 2\partial\kappa \ln(z) + \textstyle\sum_{j=1}^{K} \frac{\partial \delta_j}{z^j} &= \left(\mathfrak{a} - \rho + 2\frac{\kappa}{\rho} - \textstyle\sum_{j=1}^{K} \frac{j\delta_j}{\rho^{j+1}}\right)\partial\rho + \partial\mathfrak{b} - \partial\mathfrak{a}(\rho - \mathfrak{a}) + 2\partial\kappa \ln\rho + \textstyle\sum_{j=1}^{K} \frac{\partial\gamma_j}{\rho^j} \\
\Longrightarrow \quad \partial\rho &= \frac{-\partial f(z) + 2\partial\kappa \ln\left(\frac{z}{\rho}\right) + \sum_{j=1}^{K} \frac{\partial\delta_j}{z^j} + \partial\mathfrak{b} + \partial\mathfrak{a}(\rho - \mathfrak{a}) - \sum_{j=1}^{K} \frac{\partial\gamma_j}{\rho^j}}{\mathfrak{a} - \rho + \frac{\kappa}{\rho} - \sum_{j=1}^{K} \frac{j\gamma_j}{\rho^{j+1}}} \\
\Longrightarrow \quad \partial\rho &= \rho \frac{-\rho^K \partial f(z) + 2\rho^K \ln\left(\frac{z}{\rho}\right)\partial\kappa + \sum_{j=1}^{K} \frac{\rho^K \partial\delta_j}{z^j} + \rho^K(\rho - \mathfrak{a})\partial\mathfrak{a} - \sum_{j=1}^{K} \partial\gamma_j \rho^{K-j}}{\mathfrak{a}\rho^{K+1} - \rho^{K+2} + 2\kappa\rho^K - \sum_{j=1}^{K} j\gamma_j \rho^{K-j}}.
\end{aligned} \tag{2-23}$$

Formula (2-23) should be regarded as defining a vector field on $\mathcal{M}$, and this flow together with $\rho(z;0,\vec{0}) = \sqrt{2f(z;0,\vec{0})}$ gives an initial value problem. To see this we have to remember that the tangent space to $\mathcal{M}$ consists of

$$T\mathcal{M} := \Omega_0 \times \mathbb{C}^{K+2} \tag{2-24}$$

where $\Omega_0$ stands for the Banach vector space of bounded analytic functions on $\mathbb{D}(r)$ (without the requirement of being univalent) mapping 0 to 0. The denominator vanishes generically at $K+2$ values $\rho_j$; since $\partial\rho$ must be an analytic function, the numerator must vanish at the same points and this yields a linear system for the $K+2$ values $\partial\mathfrak{b}, \partial\mathfrak{a}, \partial\gamma_1, \ldots, \partial\gamma_K$. To see how this works in more detail, let $\rho_j$ be the roots of the denominator in (2-23)

$$-\rho^{K+2} + \mathfrak{a}\rho^{K+1} + 2\kappa\rho^K - \sum_{j=1}^{K} j\gamma_j\rho^{K-j} = -\prod_{j=1}^{K+2}(\rho - \rho_j)\ . \tag{2-25}$$

For $\kappa, \mathfrak{a}, \vec{\gamma}$ sufficiently small all the roots $\rho_j$ belong to the disk $\mathbb{D}(r)$ where $\rho(z)$ is univalent and therefore there are corresponding values $z_1, \ldots, z_{K+2}$.

The linear system that determines $\partial\mathfrak{a}, \partial\vec{\gamma}$ is then

$$\rho_\ell^K\left(-\partial f(z_\ell) + 2\ln\left(\frac{z_\ell}{\rho_\ell}\right)\partial\kappa\right) + \sum_{j=1}^{K}\frac{\rho_\ell^K}{z_\ell^j}\partial\delta_j + \rho_\ell^K(\rho_\ell - \mathfrak{a})\partial\mathfrak{a} + \rho_\ell^K\partial\mathfrak{b} + \sum_{j=1}^{K}\rho_\ell^{K-j}\partial\gamma_j = 0\ , \quad \ell = 1, \ldots, K+2. \tag{2-26}$$

What we want to see is that this system determines $\partial\mathfrak{a}, \partial\mathfrak{b}, \partial\vec{\gamma}$ as *analytic* functions of $\kappa, \mathfrak{a}, \vec{\gamma}$; to see this we observe that the coefficient matrix of the linear system (2-26) is

$$\begin{bmatrix} \rho_1^K(\rho_1 - \mathfrak{a}) & \rho_1^K & \rho_1^{K-1} & \ldots & 1 \\ \rho_2^K(\rho_2 - \mathfrak{a}) & \rho_2^K & \rho_2^{K-1} & \ldots & 1 \\ \vdots & & & & \\ \rho_{K+2}^K(\rho_{K+2} - \mathfrak{a}) & \rho_{K+2}^K & \rho_{K+2}^{K-1} & \ldots & 1 \end{bmatrix} \begin{bmatrix} \partial\mathfrak{a} \\ \partial\mathfrak{b} \\ \partial\gamma_K \\ \vdots \\ \partial\gamma_1 \end{bmatrix} = -\begin{bmatrix} H(z_1) \\ H(z_2) \\ \vdots \\ H(z_{K+2}) \end{bmatrix}, \tag{2-27}$$

$$H(z_\ell) := \rho_\ell^K\left(2\ln\left(\frac{z_\ell}{\rho_\ell}\right)\partial\kappa - \partial f(z_\ell)\right) + \sum_{j=1}^{K}\frac{\rho_\ell^K}{z_\ell^j}\partial\delta_j\ . \tag{2-28}$$

Solving this system by Cramer's rule yields $\partial\mathfrak{a}, \partial\vec{\gamma}$ as **symmetric** functions of the roots $\rho_\ell$; moreover it is seen that the determinant of the linear part is simply the Vandermonde determinant $\Delta(\vec{\rho}) := \prod_{j<\ell\leq K+2}(\rho_j - \rho_\ell)$ and since the determinants in the numerators of Cramer's formula will also vanish whenever two roots coincide, it follows that the ratio is actually *analytic* on the diagonals $\rho_\ell = \rho_k,\ \ell \neq k$.

We have thus proved that $\partial\mathfrak{a}, \partial\vec{\gamma}, \partial\mathfrak{b}$ are analytic symmetric functions of the $\rho_\ell$'s; it is well known that the ring of analytic symmetric functions is generated by the elementary symmetric polynomials in the roots, namely, the coefficients of the polynomial (2-25). This means that $\partial\mathfrak{a}, \partial\vec{\gamma}, \partial\mathfrak{b}$ are actually expressible in terms of analytic expressions of $\mathfrak{a}, \kappa, \vec{\gamma}$.

In order to complete the proof we should check that the vector field determined by (2-23) is **Lipshitz** with respect to the Banach norm of $T\mathcal{M}$; the check is rather straightforward but lengthy and a detailed analysis is deferred to App. A in the simplified case $K = 0$. After this, the existence and uniqueness of the integrated flow follows from standard theorems in Banach spaces.

**Jacobian at the origin.** To compute the Jacobian at the origin $(\kappa, \vec{\delta}) = (0, \vec{0})$ we have to set

$$\rho = \sqrt{2f(z; 0, \vec{0})}\ , \quad \mathfrak{a} = \kappa = \delta_j = \gamma_j = \mathfrak{b} = 0\ . \tag{2-29}$$

Taking now $\partial_\ell$ to mean $\partial_{\delta_\ell}$ we find the equations

$$\partial_\ell \rho = \frac{-\partial_\ell f + \frac{1}{z^\ell} + \partial_\ell \mathfrak{b} + \rho \partial_\ell \mathfrak{a} - \sum_{j=1}^K \frac{\partial_\ell \gamma_j}{\rho^j}}{-\rho}\ . \tag{2-30}$$

Since we want $\rho(0; \kappa, \vec{\delta}) \equiv 0$ we must impose that $\partial_\ell \rho$ in (2-30) vanishes at least of order $z$ at $z = 0$; this yields a linear system for the coefficients $\partial_\ell \mathfrak{b}, \partial_\ell \mathfrak{a}, \partial_\ell \gamma_j$ and in particular

$$\left\{ \begin{array}{lcl} \partial_\ell \gamma_j(0, \vec{0}) & = & 0\ , j > \ell \\ \partial_j \gamma_j(0, \vec{0}) & = & 1 \\ \partial_\ell \gamma_j(0, \vec{0}) & = & \star\ , j < \ell. \end{array} \right. \qquad \frac{\partial \vec{\gamma}}{\partial \vec{\delta}} = \begin{bmatrix} 1 & \star & \star & \dots \\ & 1 & \star & \dots \\ & & \ddots & \\ & & & 1 \end{bmatrix} \tag{2-31}$$

with the $\star$ denoting some expression which is not relevant to us; the above Jacobian is then triangular with 1 on the diagonal, and hence it is invertible in a neighborhood of $\kappa = 0,\ \vec{\delta} = \vec{0}$. $\square$

## 2.2 Determination of the $\delta_j$'s

We now introduce the rescaled variable ($\kappa > 0$)

$$\zeta = \frac{\rho}{\sqrt{\kappa}}\ , \quad \zeta_0 := \frac{\mathfrak{a}}{\sqrt{\kappa}}\ . \tag{2-32}$$

Let

$$g_H(\zeta) := -\frac{\zeta}{4}\sqrt{\zeta^2 - 4} + \ln(\zeta + \sqrt{\zeta^2 - 4}) + \frac{\zeta^2}{2} + \frac{\ell_H}{2} \tag{2-33}$$

$$\ell_H := -1 - 2\log 2 \tag{2-34}$$

be the $g$–function for the Gaussian Unitary Ensemble. It admits an asymptotic expansion of the form

$$g_H(\zeta) := \ln \zeta + \sum_{\ell=1}^{\infty} \frac{c_\ell^{(0)}}{\zeta^{2\ell}}\ . \tag{2-35}$$

We define the constants $c_j^{(H)}$ by the requirement

$$g_H(\zeta - \zeta_0) - \ln \zeta + \sum_{j}^{K} \frac{c_j^{(H)}}{\zeta^j} = \mathcal{O}(\zeta^{-K-1})\ , \quad \zeta \to \infty. \tag{2-36}$$

It is easily verified that the $c_j^{(H)}$ are polynomials in $\zeta_0$.

**Proposition 2.2.** *The parameters $\vec{\delta}$ are uniquely determined as Puiseux series of $\sqrt{\kappa}$ by the requirement*

$$\mathcal{P}_2(z) = -\frac{\kappa}{2}(\zeta - \zeta_0)^2 + 2\kappa \ln(\sqrt{\kappa}\zeta) + \kappa \sum_{j=1}^{K} \frac{\kappa^{-\frac{j}{2}}\gamma_j}{\zeta^j} + \mathfrak{b} = -\frac{\kappa}{2}(\zeta - \zeta_0)^2 + 2\kappa \ln(\sqrt{\kappa}\zeta) + \mathfrak{b} + 2\kappa \sum_{j=1}^{K} \frac{c_j^{(H)}}{\zeta^j} \tag{2-37}$$

*Moreover we have*

$$\zeta_0 = \mathcal{O}(\sqrt{\kappa}), \quad \mathfrak{b} = \mathcal{O}(\kappa)\ , \quad \vec{\delta} = \mathcal{O}(\kappa^{\frac{3}{2}}). \tag{2-38}$$

*Proof.* Recall that $\zeta_0$ depends on both $\kappa, \vec{\delta}$ analytically; although it is possible to give more detailed information about this dependence, it will not be necessary to the end of establishing the present proposition.

We need to solve the nonlinear system

$$\kappa^{-\frac{j}{2}}\gamma_j(\kappa,\vec{\delta}) = 2\kappa c_j^{(H)}(\zeta_0)\ , \quad j = 1,\dots,K \tag{2-39}$$

for the unknowns $\vec{\delta}$. The local solvability of the system around $\vec{\delta} = \vec{0}$ is guaranteed if we can guarantee the nonsingularity of the Jacobian matrix. But this system can be rewritten

$$\vec{\gamma} - 2\kappa^D \vec{c}^{(H)} = 0\ , \quad D := \mathrm{diag}\left(\frac{3}{2}, 2, \frac{5}{2}, \dots \frac{K+2}{2}\right). \tag{2-40}$$

Since $\vec{c}^{(H)}$ is analytic in $\vec{\delta}$ one promptly sees that the Jacobian is

$$J := \frac{\partial\vec{\gamma}}{\partial\vec{\delta}} - 2\kappa^D \frac{\partial\vec{c}^{(H)}}{\partial\vec{\delta}} \tag{2-41}$$

and hence $\det J = 1 + \mathcal{O}(\kappa^{\frac{3}{2}})$. This guarantees that there is a polydisk $|\kappa| < C_1,\ \|\vec{\delta}\| < C_2$ (for suitable constants) where the system admits a solution in Puiseux series (i.e. analytic in $\sqrt{\kappa}$). It is also clear from (2-40) that $\vec{\delta} = \mathcal{O}(\kappa^{\frac{3}{2}})$. Thus, $\zeta_0(\kappa, \delta(\kappa))$ is still of order $\mathcal{O}(\sqrt{\kappa})$ and $\mathfrak{b}(\kappa, \delta(\kappa))$ is still $\mathcal{O}(\kappa)$, since all these depend analytically on $\vec{\delta}$, which is not of higher order than $\kappa$. □

For future reference and definiteness we collect the result of the above discussion in the theorem below

**Theorem 2.2.** *There exists a conformal change of coordinate $\zeta(z;\kappa)$ of the form*

$$\zeta(z;\kappa) = \frac{\rho(z;\kappa)}{\sqrt{\kappa}} = \frac{1}{\sqrt{\kappa}} C(z - a^\star)(1 + \mathcal{O}(z - a^\star))\ , C > 0 \tag{2-42}$$

*and a choice of $\vec{\delta} = \vec{\delta}(\kappa)$ for the deformed potential (2-2) in Puiseux series of $\sqrt{\kappa}$ such that*

$$\mathcal{P}_2(z;\kappa,\vec{\delta}(\kappa)) = -\frac{\kappa}{2}(\zeta - \zeta_0)^2 + 2\kappa \ln(\sqrt{\kappa}\zeta) + \mathfrak{b} + 2\kappa \sum_{j=1}^{K} \frac{c^{(H)}}{\zeta^j}\ . \tag{2-43}$$

*The functions $\zeta_0(\kappa), \beta(\kappa), \vec{\delta}(\kappa)$ admit a Puiseux expansion and are of orders*

$$\zeta_0 = \mathcal{O}(\sqrt{\kappa})\ , \quad \beta = \mathcal{O}(\kappa)\ , \quad \vec{\delta} = \mathcal{O}(\kappa). \tag{2-44}$$

*The expressions $c_j^{(H)}$ are polynomials of degree $j$ in $\zeta_0$ determined by the formula* (2-36).

### 2.3 Steepest descent analysis (supercritical case)

We make the following change of variables from $\mathbf{Y}(z)$ to $\mathbf{W}(z)$:

$$\mathbf{W}(z;\kappa) \quad := \begin{pmatrix} e^{\frac{n}{2}\ell_1} & 0 & 0 \\ 0 & e^{-\frac{n}{2}\ell_1} & 0 \\ 0 & 0 & e^{\frac{n}{2}(\ell_1 - 2l_2 + 2\eta)} \end{pmatrix} \mathbf{Y}(z) \begin{pmatrix} e^{-\frac{n}{2}V} & 0 & 0 \\ 0 & e^{\frac{n}{2}V} & 0 \\ 0 & 0 & e^{\frac{n}{2}(V-2az)} \end{pmatrix} \times \tag{2-45}$$

$$\times \begin{Bmatrix} \mathbf{I}, & z \in \Omega_1 \cup \Omega_6 \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Omega_2 \cup \Omega_5 \\ \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Omega_3 \\ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Omega_4 \end{Bmatrix} \begin{pmatrix} e^{-\frac{n}{2}\mathcal{P}_1} & 0 & 0 \\ 0 & e^{\frac{n}{2}\mathcal{P}_1} & 0 \\ 0 & 0 & e^{\frac{n}{2}(2\mathcal{P}_2 - \mathcal{P}_1 - 2\eta)} \end{pmatrix} e^{-\frac{n}{2}\delta V(z)} \tag{2-46}$$

$$\delta V(z) := \kappa \ln(z - a^\star) + \kappa \sum_{j=1}^{K} \frac{\delta_j}{2(z-a^\star)^j} \tag{2-47}$$

$$\eta := \kappa \ln \kappa + \mathfrak{b} + \kappa \ell_H \ , \quad (\ell_H := -1 - 2\ln 2). \tag{2-48}$$

The constant (in $z$) $\eta$ is chosen to carefully balance other constants later on in the study of the local parametrix; we recall that $\mathfrak{b} = \mathcal{O}(\kappa)$ has appeared in Theorem 2.2, which is also the source of the $\kappa \ln \kappa$ term. The constant $\ell_H$ was introduced in (2-34) and it is the Robin's constant for the equilibrium problem associated to the quadratic potential. See Figure 1 for a visual on the different regions $\Omega_j$'s. The exact choice of the outer lenses is given below in the proof of Lemma 2.1(d). The inner lenses are chosen in the standard way for the $2 \times 2$ Riemann-Hilbert problem for (non-multiple) orthogonal polynomials.

The new matrix $\mathbf{W}(z)$ satisfies a new Riemann Hilbert Problem which can be directly evinced from the one for $\mathbf{Y}$ and is of the form $\mathbf{W}_+(z) = \mathbf{W}_-(z)\mathbf{V}^{(\mathbf{W})}(z)$ with jumps

$$\mathbf{V}^{(\mathbf{W})}(z) = \begin{cases} \begin{pmatrix} 1 & e^{n\mathcal{P}_1(z)} & e^{n(\mathcal{P}_2(z)-\eta)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \partial\Omega_1 \cap \partial\Omega_6, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -e^{n(\mathcal{P}_3(z)-\eta)} \\ 0 & 0 & 1 \end{pmatrix}, & z \in (\partial\Omega_1 \cap \partial\Omega_2) \cup (\partial\Omega_5 \cap \partial\Omega_6), \\ \begin{pmatrix} 1 & 0 & 0 \\ e^{-n\mathcal{P}_1(z)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in (\partial\Omega_2 \cap \partial\Omega_3) \cup (\partial\Omega_4 \cap \partial\Omega_5), \\ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in [\alpha, \beta], \\ \begin{pmatrix} 1 & e^{n\mathcal{P}_1(z)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \partial\Omega_2 \cap \partial\Omega_5, \end{cases} \tag{2-49}$$

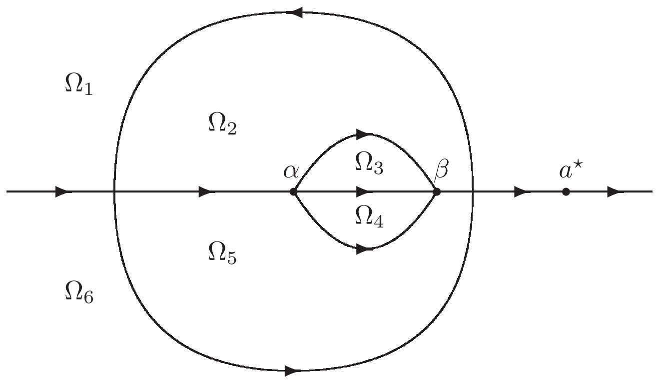


Figure 1: The oriented jump contour $\Gamma$ for $\mathbf{W}(z)$ and the regions $\Omega_i$ in the supercritical case.

and the asymptotic conditions

$$\mathbf{W}(z) = \mathbf{I} + \mathcal{O}\left(\frac{1}{z}\right), \quad z \to \infty \tag{2-50}$$

$$\mathbf{W}(z) = (\text{analytic}) \begin{pmatrix} \mathrm{e}^{-n\delta V(z)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{e}^{n\delta V(z)} \end{pmatrix} \quad \text{as } z \to a^\star. \tag{2-51}$$

The orientation of the contours is given in Figure 1. Here we have used the factorization

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2-52}$$

### 2.3.1 The outer parametrix

We will show below in Section 2.4 that the jump matrices for $\mathbf{W}(z)$ decay uniformly to constant jump matrices as $n \to \infty$ outside of small fixed neighborhoods of $\alpha$, $\beta$, and $a^\star$. These limiting jump matrices are the identity except on the band $[\alpha, \beta]$. We therefore define the outer parametrix $\mathbf{\Psi}(z)$ to be the solution to the following Riemann-Hilbert problem:

$$\mathbf{\Psi}_+(z) = \mathbf{\Psi}_-(z) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ for } z \in (\alpha, \beta)\ , \qquad \mathbf{\Psi}(z) = \mathbf{I} + \mathcal{O}\left(\frac{1}{z}\right). \tag{2-53}$$

It is well known that the solution to this Riemann-Hilbert problem is

$$\mathbf{\Psi}(z) = \mathbf{U}^{-1} \begin{pmatrix} \left(\dfrac{z-\beta}{z-\alpha}\right)^{-1/4} & 0 & 0 \\ 0 & \left(\dfrac{z-\beta}{z-\alpha}\right)^{1/4} & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{U}, \quad \mathbf{U} := \begin{pmatrix} \frac{1}{2} & \frac{i}{2} & 0 \\ -\frac{1}{2} & \frac{i}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2-54}$$

where

$$\lim_{z\to\infty}\left(\frac{z-\beta}{z-\alpha}\right)^{1/4} = 1 \tag{2-55}$$

and this function is cut along $[\alpha,\beta]$.

### 2.3.2 The local parametrix near $a^\star$

Special attention is needed near the point $z = a^\star$ as the jump matrices do not decay uniformly to the identity near this point. According to the definition of $a^\star$ as the point of maximum for $P_2$ and given that $\mathcal{P}_2$ is a deformation of $P_2$ we have

$$\mathcal{P}_2(z;\kappa) = -\overbrace{\frac{C}{2}(z-a^\star)^2(1+\mathcal{O}(z-a^\star)) + \mathcal{O}(\delta)}^{:=f(z;\kappa)} + 2\kappa\log(z-a^\star) + \sum_{j=1}^{2k}\frac{\delta_j(\kappa)}{(z-a^\star)^j} \tag{2-56}$$

where $C > 0$ and the deformation $\mathcal{O}(\delta)$ is some analytic function of $z$ of the indicated order in $\kappa$. Let $\mathbb{D}_{a^\star}$ be a fixed-size circular disk centered at $a^\star$ chosen small enough so that

$$\left|\Re\left[\frac{f(z;\kappa)}{2}\right]\right| < |\Re P_1(z)| \tag{2-57}$$

(recall that $\Re P_1 > 0$ outside of the support of the equilibrium measure) inside the disk, and such that the disk does not intersect the outer lenses. Orient $\partial\mathbb{D}_{a^\star}$ clockwise.

We now apply Theorem 2.2: let the $k$ constants $c_j^{(H)}$, $j = 1,\dots,k$ be specified by (2-36): in the local scaling coordinate $\zeta$ we have

$$n\mathcal{P}_2 = -\frac{r}{2}(\zeta-\zeta_0)^2 + 2r\ln\zeta + 2r\sum_{j=1}^{K}\frac{c^{(H)}}{\zeta^j} + r\ln\kappa + n\mathfrak{b}. \tag{2-58}$$

where the scaling coordinate $\zeta$ has the following behavior on the boundary of the disk $\mathbb{D}_{a^\star}$

$$\zeta = \mathcal{O}(n^{(1-\gamma)/2}) \quad \text{when } z\in\partial\mathbb{D}_{a^\star}. \tag{2-59}$$

We also recall (Theorem 2.2) that $\mathfrak{b} = \mathcal{O}(\kappa)$ and hence $n\mathfrak{b} = \mathcal{O}(r)$. This suggests the following definition for the model Riemann Hilbert Problem of the local parametrix.

**Definition 2.1.** *The local parametrix within the disk $\mathbb{D}_{a^\star}$ shall be the unique solution $\mathbf{R}(z)$ to the following* *model Riemann-Hilbert:*

$$
\begin{cases}
\mathbf{R}_+(\zeta) = \mathbf{R}_-(\zeta) \begin{pmatrix} 1 & 0 & \zeta^{2r} \exp\left(-\frac{r}{2}(\zeta-\zeta_0)^2 + r\ell_H + 2r\sum_{j=1}^K \frac{c_j^{(H)}}{\zeta^j}\right) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\qquad\qquad = \mathbf{R}_-(\zeta) \begin{pmatrix} 1 & 0 & e^{n(\mathcal{P}_2-\eta)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \zeta \in \mathbb{R}, \\
\mathbf{R}(\zeta) = \mathbf{I} + \mathcal{O}\left(\frac{1}{\zeta}\right) \ \textit{as}\ \zeta \to \infty, \\
\mathbf{R}(\zeta) = (\textit{analytic}) \begin{pmatrix} \zeta^{-r}\exp\left(-r\sum_{j=1}^K \frac{c_j^{(H)}}{\zeta^j}\right) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta^r \exp\left(r\sum_{j=1}^K \frac{c_j^{(H)}}{\zeta^j}\right) \end{pmatrix} \ \textit{as}\ \zeta \to 0.
\end{cases} \tag{2-60}
$$

$$
\eta := \kappa \ln\kappa + \mathfrak{b} - \kappa\ell_H \tag{2-61}
$$

The behavior at $\zeta = 0$ ($z = a^\star$) is dictated by (2-51). We point out that the problem is essentially $2\times 2$; moreover it will be shown below that it is a slight modification of the Fokas-Its-Kitaev Riemann-Hilbert problem for Hermite orthogonal polynomials (see [27] and [24], Section 3).

**Proposition 2.3.** *Let the rescaled Hermite polynomials $H_i^{(r)}(\zeta)$ be the family of monic polynomials satisfying the orthogonality condition*

$$
\int_{-\infty}^{\infty} H_i^{(r)}(\zeta) H_j^{(r)}(\zeta) e^{-\frac{r}{2}\zeta^2} d\zeta = r^{j-\frac{1}{2}} j! \sqrt{2\pi}\delta_{ij} = k_j^{(r)}\delta_{ij}, \tag{2-62}
$$

*where the $k_i^{(r)}$ are normalization constants. Then the solution to* (2-60) *is*

$$
\mathbf{R}(\zeta) = \exp\left(-\frac{r}{2}\ell_H \mathbf{\Lambda}_{13}\right) \mathbf{H}_{13}(\zeta)\, \zeta^{-r\mathbf{\Lambda}_{13}}\, \exp\left(\left(\frac{r}{2}\ell_H - r\sum_{j=1}^K \frac{c_j^{(H)}}{\zeta^j}\right)\mathbf{\Lambda}_{13}\right), \tag{2-63}
$$

*where*

$$
\mathbf{H}_{13}(\zeta) := \begin{pmatrix} H_r^{(r)}(\zeta-\zeta_0) & 0 & \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{H_r^{(r)}(s-\zeta_0)e^{-\frac{r}{2}s^2}}{s-\zeta} ds \\ 0 & 1 & 0 \\ \frac{2\pi i}{-k_{r-1}^{(r)}} H_{r-1}^{(r)}(\zeta-\zeta_0) & 0 & \frac{-1}{k_{r-1}^{(r)}}\int_{-\infty}^{\infty} \frac{H_{r-1}^{(r)}(s-\zeta_0)e^{-\frac{r}{2}s^2}}{s-\zeta} ds \end{pmatrix} \tag{2-64}
$$

*and*

$$
\mathbf{\Lambda}_{13} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2-65}
$$

The proof is a direct manipulation and it is left to the reader.

Now the well known asymptotics of Hermite polynomials can be written as a **joint** asymptotic expansion for $\zeta$ **and** $r$ large as follows

$$\mathbf{H}_{13}(\zeta) = e^{\frac{r}{2}\ell_H \boldsymbol{\Lambda}_{13}} \left(\mathbf{I} + \mathcal{O}\left(\frac{1}{r(|\zeta|+1)}\right)\right)\mathbf{U}^{-1} \begin{pmatrix} \sqrt[4]{\frac{\zeta-\zeta_0-2}{\zeta-\zeta_0+2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt[4]{\frac{\zeta-\zeta_0+2}{\zeta-\zeta_0-2}} \end{pmatrix} \mathbf{U} e^{r g_H(\zeta-\zeta_0)\boldsymbol{\Lambda}_{13} - \frac{r}{2}\ell_H \boldsymbol{\Lambda}_{13}} \tag{2-66}$$

$$\mathbf{U} := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ -1 & 0 & i \end{bmatrix} \tag{2-67}$$

and $g_H(\zeta)$ is given by (2-33). For large $\zeta$, we have the expansion of $g_H$ as in (2-36). The error term of $\mathcal{O}(1/r(|\zeta|+1))$ in (2-66) is from the standard Airy parametrix used in the Riemann-Hilbert problem for Hermite orthogonal polynomials at $\zeta = \pm 2$. For more details on this calculation see equation (7.72) in [21] or equation (4.16) and Appendix B in [24], noting that the variable $\zeta$ is rescaled by a constant factor. Then for large $\zeta$ (such as on $\partial\mathbb{D}_{a^\star}$) we can estimate

$$\begin{aligned} \mathbf{R}(\zeta) &= \left(\mathbf{I} + \mathcal{O}\left(\frac{1}{r\zeta}\right)\right)\left(\mathbf{I} + \mathcal{O}\left(\frac{1}{\zeta}\right)\right) \exp\left[r\left(g_H(\zeta-\zeta_0) - \log\zeta - \sum_{j=1}^{K} \frac{c_j^{(H)}}{\zeta^j}\right)\boldsymbol{\Lambda}_{13}\right] \\ &= \mathbf{I} + \mathcal{O}\left(\frac{1}{\zeta}\right) + \mathcal{O}\left(\frac{r}{\zeta^{K+1}}\right). \end{aligned} \tag{2-68}$$

## 2.4 Error analysis in the supercritical case

Let $\mathbb{D}_\alpha$ and $\mathbb{D}_\beta$ be small, fixed, closed disks centered at $\alpha$ and $\beta$ that are bounded away from the outer lenses. Orient the boundaries $\partial\mathbb{D}_\alpha$ and $\partial\mathbb{D}_\beta$ clockwise. For $z \in \mathbb{D}_\alpha$, let $\mathbf{P}_{\mathrm{Ai}}^{(\alpha)}(z)$ be the *Airy parametrix* satisfying

- $\mathbf{P}_{\mathrm{Ai}}^{(\alpha)}(z)$ has the same jumps as $\mathbf{W}(z)$ for $z \in \mathbb{D}_\alpha$,
- $\mathbf{P}_{\mathrm{Ai}}^{(\alpha)}(z)\boldsymbol{\Psi}(z)^{-1} = \mathbf{I} + \mathcal{O}\left(\frac{1}{n}\right)$ for $z \in \mathbb{D}_\alpha$.

The construction of the Airy parametrix is standard, involving Airy functions and a local change of variables. See [14] Section 7, for example, for an Airy parametrix for a $3 \times 3$ Riemann-Hilbert problem. The Airy parametrix $\mathbf{P}_{\mathrm{Ai}}^{(\beta)}(z)$ is defined analogously for $z \in \mathbb{D}_\beta$.

We now define the global parametrix $\boldsymbol{\Psi}^\infty(z)$ by

$$\boldsymbol{\Psi}^\infty(z) := \begin{cases} \boldsymbol{\Psi}(z), & z \notin \mathbb{D}_\alpha \cup \mathbb{D}_\beta \cup \mathbb{D}_{a^\star}, \\ \boldsymbol{\Psi}(z)\mathbf{R}(\zeta(z)), & z \in \mathbb{D}_{a^\star}, \\ \mathbf{P}_{\mathrm{Ai}}^{(\alpha)}(z), & z \in \mathbb{D}_\alpha \\ \mathbf{P}_{\mathrm{Ai}}^{(\beta)}(z), & z \in \mathbb{D}_\beta. \end{cases} \tag{2-69}$$

The *error matrix* $\mathbf{E}(z)$ is given by

$$\mathbf{E}(z) := \mathbf{W}(z)\boldsymbol{\Psi}^\infty(z)^{-1}. \tag{2-70}$$

Let $\Gamma$ denote the contours given by the boundaries of the regions $\Omega_j$ in Figure 1. The error matrix satisfies a Riemann-Hilbert problem with the following jumps:

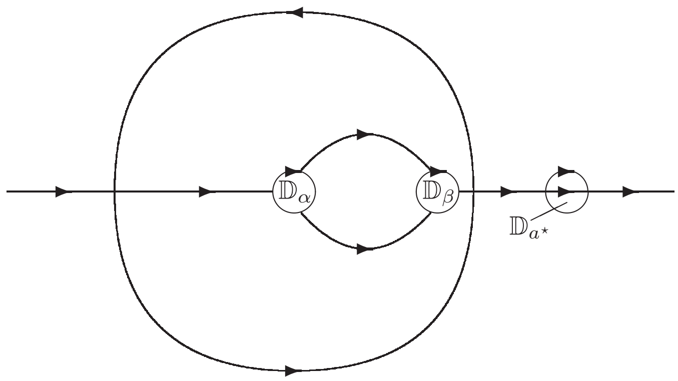


Figure 2: The jump contours $\Gamma^{(\mathbf{E})}$ for the Riemann-Hilbert problem for $\mathbf{E}(z)$ in the supercritical case.

- For $z$ outside the disks $\mathbb{D}_\alpha$, $\mathbb{D}_\beta$, and $\mathbb{D}_{a^\star}$, and excluding the band $[\alpha, \beta]$:

$$\mathbf{V}^{(\mathbf{E})}(z) = \mathbf{\Psi}(z)\mathbf{V}^{(\mathbf{W})}(z)\mathbf{\Psi}(z)^{-1}, \quad z \in \Gamma \cap (\mathbb{D}_\alpha \cup \mathbb{D}_\beta \cup \mathbb{D}_{a^\star})^c \cap [\alpha, \beta]^c, \tag{2-71}$$

where $\mathbf{V}^{(\mathbf{W})}(z)$ is given by (2-49).

- For $z$ on the boundaries of the disks $\partial\mathbb{D}_\alpha$, $\partial\mathbb{D}_\beta$, and $\partial\mathbb{D}_{a^\star}$:

$$\mathbf{V}^{(\mathbf{E})}(z) = \begin{cases} \mathbf{\Psi}(z)\mathbf{R}(\zeta)\mathbf{\Psi}(z)^{-1}, & z \in \partial\mathbb{D}_{a^\star}, \\ \mathbf{P}_{\mathrm{Ai}}^{(\alpha)}(z)\mathbf{\Psi}(z)^{-1}, & z \in \partial\mathbb{D}_\alpha, \\ \mathbf{P}_{\mathrm{Ai}}^{(\beta)}(z)\mathbf{\Psi}(z)^{-1}, & z \in \partial\mathbb{D}_\beta. \end{cases} \tag{2-72}$$

- For $z$ inside the disk $\mathbb{D}_{a^\star}$:

$$\mathbf{V}^{(\mathbf{E})}(z) = \mathbf{\Psi}(z)\mathbf{R}_-(\zeta)\begin{pmatrix} 1 & e^{n\mathcal{P}_1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\mathbf{R}_-(\zeta)^{-1}\mathbf{\Psi}(z)^{-1}, \quad z \in \Gamma \cap \mathbb{D}_{a^\star}. \tag{2-73}$$

- Furthermore, $\mathbf{V}^{(\mathbf{E})}(z) = \mathbf{I}$ on the contours

$$[\alpha, \beta] \cap (\mathbb{D}_\alpha \cup \mathbb{D}_\beta)^c, \quad \Gamma \cap \mathbb{D}_\alpha, \quad \text{and} \quad \Gamma \cap \mathbb{D}_\beta.$$

The jump contours $\Gamma^{(\mathbf{E})}$ for $\mathbf{E}(z)$ are shown in Figure 2.

We now show that all of the jump matrices in (2-71)-(2-73) are uniformly close to the identity as $n \to \infty$.

For the error bounds it will be convenient to split $\Gamma^{(\mathbf{E})}$ into a compact component $\Gamma_C^{(\mathbf{E})}$ and a noncompact component $\Gamma_N^{(\mathbf{E})}$:

$$\begin{aligned} \Gamma_C^{(\mathbf{E})} &:= \partial\mathbb{D}_\alpha \cup \partial\mathbb{D}_\beta \cup \partial\mathbb{D}_{a^\star} \cup (\Gamma \cap \mathbb{D}_{a^\star}), \\ \Gamma_N^{(\mathbf{E})} &:= \Gamma^{(\mathbf{E})} \backslash \Gamma_C^{(\mathbf{E})}. \end{aligned} \tag{2-74}$$

We now gather the results we will need on the functions $P_1(z)$, $P_2(z)$, and $P_3(z)$ defined by (1-6)–(1-8).

**Lemma 2.1.** *In the supercritical regime, the inner and outer lenses can be chosen so that:*

(a) *On the inner lenses outside of the disks around $\alpha$ and $\beta$: The real part of $P_1(z)$ is positive and bounded away from zero for $z \in [(\partial\Omega_2 \cap \partial\Omega_3) \cup (\partial\Omega_4 \cap \partial\Omega_5)] \cap (\mathbb{D}_\alpha \cup \mathbb{D}_\beta)^c$.*

(b) *On the real axis outside of $[\alpha, \beta]$ and the disks around $\alpha$ and $\beta$: The real part of $P_1(z)$ is negative and bounded away from zero for $z \in [(\partial\Omega_1 \cap \partial\Omega_6) \cup (\partial\Omega_2 \cap \partial\Omega_5)] \cap (\mathbb{D}_\alpha \cup \mathbb{D}_\beta)^c$.*

(c) *On the real axis outside of the band $[\alpha, \beta]$ and a fixed distance away from $\alpha$, $\beta$, and $a^\star$: The real part of $P_2(z)$ is negative and bounded away from zero for $z \in [(\partial\Omega_2 \cap \partial\Omega_5) \cap (\mathbb{D}_\alpha \cup \mathbb{D}_\beta \cup \mathbb{D}_{a^\star})^c] \cup (\partial\Omega_1 \cap \partial\Omega_6)$.*

(d) *On the outer lenses: For $\kappa$ sufficiently small, the real part of $P_3(z)$ is negative and bounded away from zero for $z \in (\partial\Omega_1 \cap \partial\Omega_2) \cup (\partial\Omega_5 \cap \partial\Omega_6)$.*

*Proof.* Statements (a) and (b) follow from the analysis of the Riemann-Hilbert problem for the standard (not multiple) orthogonal polynomials (see, for instance, [21]). Statement (c) follows from the definitions of the supercritical region and $a^\star$.

For (d), we begin by choosing the outer lenses used to define $\mathbf{W}(z)$. Fix $\kappa = 0$. Note that

$$P_3(\beta) = -P_1(\beta) + P_2(\beta) = P_2(\beta) < P_2(a^\star) = 0. \tag{2-75}$$

The second equality uses the fact that $\Re P_1$ is zero on $[\alpha, \beta]$ and $P_1$ is real for $x > \beta$; the inequality follows since $a^\star$ is the location of the global maximum of $P_2(z)$; and the final equality is true by the choice of the Lagrange multiplier $l_2$. Thus, there is a fixed radius neighborhood of $\beta$ in which $\mathrm{Re} P_3(z) < 0$ for real $z$. We choose the outer lenses to be a circle centered below $\alpha$ whose right-most endpoint passes through the real axis at some point on $(\beta, a^\star)$. We choose the circle big enough such that $\Re P_2$ is negative on the real axis to the left of the circle. This is always possible due to Assumption 1.1 (iv).

We now show that the outer lenses are descent lines of $\mathrm{Re}\, P_3(z)$. Clearly the real part of $az$ decreases as we move to the left along the lenses. Note that $\mathrm{Re}\, g(z) = \int_\alpha^\beta \log|z - s|\rho_{\min}(s) ds$ where $\rho_{\min}(s)$ is the associated equilibrium measure. Now for any $s \in (\alpha, \beta)$, $\log|z - s|$ is increasing as $z$ moves to the left along the lens (one can see this clearly by drawing a circle that is centered at $s$ and is tangent to the outer lenses at the right-most point). Since $\rho_{\min}(s)$ is positive for $s \in (\alpha, \beta)$, $\mathrm{Re}\, g(z)$ increases as $z$ moves to the left along the lenses. This shows (d) for $\kappa = 0$. Since $\widetilde{\alpha}(\kappa) = \alpha + \mathcal{O}(\kappa)$ and $\widetilde{\beta} = \beta + \mathcal{O}(\kappa)$, (d) also holds for $\kappa$ sufficiently small. □

We now present the results we will need for the modified functions $\mathcal{P}_1(z;\kappa)$, $\mathcal{P}_2(z;\kappa)$, and $\mathcal{P}_3(z;\kappa)$ defined by (2-12)–(2-14).

**Lemma 2.2.** *For $\kappa$ sufficiently small (refer to Figure 1 and Figure 2):*

(a) *On the inner lenses outside of the disks around $\alpha$ and $\beta$: The real part of $\mathcal{P}_1(z;\kappa)$ is positive and bounded away from zero for $z \in [(\partial\Omega_2 \cap \partial\Omega_3) \cup (\partial\Omega_4 \cap \partial\Omega_5)] \cap (\mathbb{D}_\alpha \cup \mathbb{D}_\beta)^c$.*

(b) *On the real axis outside of $[\alpha, \beta]$ and a fixed distance away from $\alpha$, $\beta$, and $a^\star$: The real part of $\mathcal{P}_1(z;\kappa)$ is negative and bounded away from zero for $z \in [(\partial\Omega_1 \cap \partial\Omega_6) \cup (\partial\Omega_2 \cap \partial\Omega_5)] \cap (\mathbb{D}_\alpha \cup \mathbb{D}_\beta \cup \mathbb{D}_{a^\star})^c$.*

*(c) On the real axis outside of the outer lenses and a fixed distance away from $a^\star$: The real part of $\mathcal{P}_2(z;\kappa)$ is negative and bounded away from zero for $z \in (\partial\Omega_1 \cap \partial\Omega_6) \cap (\mathbb{D}_{a^\star})^c$.*

*(d) On the outer lenses: The real part of $\mathcal{P}_3(z;\kappa)$ is negative and bounded away from zero for $z \in (\partial\Omega_1 \cap \partial\Omega_2) \cup (\partial\Omega_5 \cap \partial\Omega_6)$.*

*(e) For real $z$ inside $\mathbb{D}_{a^\star}$: Let $g_H(\zeta)$ be defined by (2-33). Then the real part of*

$$\mathcal{P}_1(z;\kappa) + \kappa g_H(\zeta) - \kappa \log\zeta - \kappa \sum_{j=1}^{K} \frac{c_j^{(H)}}{\zeta^j}$$

*is negative and bounded away from zero for $z \in \Gamma \cap \mathbb{D}_{a^\star}$.*

*Proof.* Parts (a) through (d) follow from Lemma 2.1(a)–(d) together with the convergence $\mathfrak{g} \to g$ guaranteed in Proposition 2.1, the boundedness of $\log(z-a^\star)$ and $(z-a^\star)^{-j}$, $j = 1, \ldots, K$ outside of $\mathbb{D}_{a^\star}$, and $\delta_j(0) = 0$.

For part (e), first note that comparing the two expressions (2-13) and (2-58) for $\mathcal{P}_2(z;\kappa)$ gives

$$\kappa\log(z-a^\star) - \kappa\log\zeta + \sum_{j=1}^{K} \frac{\delta_j}{2(z-a^\star)^j} - \kappa\sum_{j=1}^{K}\frac{c_j^{(H)}}{\zeta^j} = \frac{1}{2}f(z;\kappa) - \frac{1}{2}\mathfrak{b} - \kappa\ln\sqrt{\kappa} - \frac{\kappa}{4}(\zeta-\zeta_0)^2. \tag{2-76}$$

By the choice of $\mathbb{D}_{a^\star}$ (see (2-57)), for $\kappa$ sufficiently small we have

$$\left|\Re\left[\frac{1}{2}f(z;\kappa) - \frac{\mathfrak{b}}{2} - \kappa\ln\sqrt{\kappa} - \frac{\kappa}{4}(\zeta-\zeta_0)^2\right]\right| < |\Re P_1(z)| \text{ for } z \in \mathbb{D}_{a^\star}. \tag{2-77}$$

Write

$$\begin{aligned} &\mathcal{P}_1(z;\kappa) + \kappa g_H(\zeta) - \kappa\log\zeta - \kappa\sum_{j=1}^{k}\frac{c_j^{(H)}}{\zeta^{2j}} \\ &\quad = \underbrace{-V + 2(1-\kappa)\mathfrak{g} + \ell_1 + \kappa g_H}_{=P_1+\mathcal{O}(\kappa)} + \underbrace{\kappa\log(z-a^\star) - \kappa\log\zeta + \sum_{j=1}^{K}\frac{\delta_j}{2(z-a^\star)^j} - \kappa\sum_{j=1}^{K}\frac{c_j^{(H)}}{\zeta^j}}_{\text{Has real part bounded above by } |\Re P_1|}. \end{aligned} \tag{2-78}$$

For $\kappa$ small, by the convergence $\mathfrak{g} \to g$ in Proposition 2.1, the first group of terms on the left-hand side, $-V + 2(1-\kappa)\mathfrak{g} + \ell_1 + \kappa g_H$, is within $\mathcal{O}(\kappa)$ of $P_1$, which has strictly negative real part for $z \in \mathbb{D}_{a^\star}$. Since the real part of the second group of terms is strictly less than the magnitude of the real part of $P_1$, part (e) follows. ∎

We are now in a position to bound the jumps $\mathbf{V}^{(\mathbf{E})}(z)$ of the error problem.

**Lemma 2.3.** *In the supercritical regime, for large $n$,*

*(a) Outside the disks $\mathbb{D}_\alpha$, $\mathbb{D}_\beta$, and $\mathbb{D}_{a^\star}$: There is a constant $c > 0$ such that*

$$\mathbf{V}^{(\mathbf{E})}(z;\kappa) = \mathbf{I} + \mathcal{O}(e^{-cn}), \quad z \in \Gamma_N^{(\mathbf{E})}.$$

(b) *On the boundary of* $\mathbb{D}_{a^\star}$*:*

$$\mathbf{V}^{(\mathbf{E})}(z;\kappa) = \mathbf{I} + \mathcal{O}\left(n^{-(1-\gamma)/2}\right) + \mathcal{O}\left(n^{\gamma - \frac{1-\gamma}{2}(K+1)}\right), \quad z \in \partial\mathbb{D}_{a^\star}.$$

(c) *On the boundaries of* $\mathbb{D}_\alpha$ *and* $\mathbb{D}_\beta$*:*

$$\mathbf{V}^{(\mathbf{E})}(z;\kappa) = \mathbf{I} + \mathcal{O}\left(n^{-1}\right), \quad z \in \partial\mathbb{D}_\alpha \cup \partial\mathbb{D}_\beta.$$

(d) *Inside* $\mathbb{D}_{a^\star}$*:*

$$\mathbf{V}^{(\mathbf{E})}(z) = \mathbf{I} + \mathcal{O}(e^{-cn}), \quad z \in \Gamma \cap \mathbb{D}_{a^\star}.$$

*Proof.* Part (a) follows from (2-71), Lemma 2.2(a)–(d), and the boundedness of $\mathbf{\Psi}(z)$. Part (b) follows from (2-59), (2-68), and the boundedness of $\mathbf{\Psi}(z)$. Part (c) comes from the construction of the parametrices $\mathbf{P}_{\mathbf{Ai}}^{(\alpha)}(z)$ and $\mathbf{P}_{\mathbf{Ai}}^{(\beta)}(z)$ (see, for instance, [24]).

For part (d) we consider the jumps (2-73) inside the disk $\mathbb{D}_{a^\star}$. Looking at the formula (2-63) for $\mathbf{R}(\zeta)$, it appears there may be a problem at $\zeta = 0$. However, note that

$$\begin{aligned}\mathbf{V}^{(\mathbf{E})}(z;\kappa) =& \Psi(z) e^{-\frac{r}{2}\ell_H \mathbf{\Lambda}_{13}} \mathbf{H}_{13-}(\zeta) e^{-r(g_H(\zeta) - \frac{\ell_H}{2})\mathbf{\Lambda}_{13}} \begin{pmatrix} 1 & (*)_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \\ & \times e^{r(g_H(\zeta) - \frac{\ell_H}{2})\mathbf{\Lambda}_{13}} \mathbf{H}_{13-}(\zeta)^{-1} e^{\frac{r}{2}\ell_H \mathbf{\Lambda}_{13}} \Psi(z)^{-1}, \quad z \in \Gamma \cap \mathbb{D}_{a^\star},\end{aligned} \tag{2-79}$$

wherein

$$(*)_{12} = \exp\left(n\mathcal{P}_1(z;\kappa) + rg_H(\zeta) - r\log\zeta - r\sum_{j=1}^{K} \frac{c_j^{(H)}}{\zeta^j}\right). \tag{2-80}$$

Now (2-79) together with Lemma 2.2(e) and the boundedness of $\mathbf{\Psi}(z)$ inside $\mathbb{D}_{a^\star}$ establishes (d). □

We can now show that the error matrix $\mathbf{E}(z)$ is uniformly close to the identity.

**Lemma 2.4.** *In the supercritical regime, for $n$ large,*

$$\mathbf{E}(z) = \mathbf{I} + \mathcal{O}\left(n^{-(1-\gamma)/2}\right)$$

*uniformly in $z$.*

*Proof.* From Lemma 2.3(b)–(d),

$$\mathbf{V}^{(\mathbf{E})}(z) = \mathbf{I} + \mathcal{O}\left(n^{-(1-\gamma)/2}\right) + \mathcal{O}\left(n^{\gamma - \frac{1-\gamma}{2}(K+1)}\right), \quad z \in \Gamma_C^{(\mathbf{E})}. \tag{2-81}$$

The first error term always dominates or matches the second term if the nonnegative integer $K$ is chosen so (2-1) is satisfied. Then, for $n$ sufficiently large there exists a constant $c$ such that

$$||\mathbf{V}^{(\mathbf{E})} - \mathbf{I}||_{L^2\left(\Gamma_C^{(\mathbf{E})}\right)} + ||\mathbf{V}^{(\mathbf{E})} - \mathbf{I}||_{L^\infty\left(\Gamma_C^{(\mathbf{E})}\right)} \le cn^{-(1-\gamma)/2}. \tag{2-82}$$

Also, from Lemma 2.3(a), for $n$ sufficiently large there is a constant $c$ such that

$$||\mathbf{V}^{(\mathbf{E})} - \mathbf{I}||_{L^2\left(\Gamma_N^{(\mathbf{E})}\right)} + ||\mathbf{V}^{(\mathbf{E})} - \mathbf{I}||_{L^\infty\left(\Gamma_N^{(\mathbf{E})}\right)} \le ce^{-cn}, \tag{2-83}$$

The result follows by a standard technique that consists of writing the solution to the Riemann-Hilbert problem in terms of a Neumann series involving $\mathbf{V}^{(\mathbf{E})} - \mathbf{I}$ (see, for instance, [24] Section 7.2 or [26] Section 3.5). □

## 2.5 The supercritical kernel and proof of Theorem 1.1

*Proof of Theorem 1.1.* Recall that the kernel is defined by (1-18):

$$K_n(x,y) = \frac{e^{-\frac{1}{2}n(V(x)+V(y))}}{2\pi i(x-y)}\left(\left[\mathbf{Y}(y)^{-1}\mathbf{Y}(x)\right]_{21} + e^{nay}\left[\mathbf{Y}(y)^{-1}\mathbf{Y}(x)\right]_{31}\right). \tag{2-84}$$

We consider local coordinates $\zeta_x$ and $\zeta_y$ in $\mathbb{D}_{a^\star}$. While the function $\mathbf{Y}(z)$ has a jump in this region, the first column of $\mathbf{Y}(z)$ does not (see the Riemann-Hilbert problem (1-17)). Therefore we can pick $x$ and $y$ to be in a convenient region. We choose $x$ and $y$ to be in $\Omega_1$ as defined in Figure 1.

From the transformation (2-45), we see that

$$\left[\mathbf{Y}(y)^{-1}\mathbf{Y}(x)\right]_{21} = \left[\mathbf{W}(y)^{-1}\mathbf{W}(x)\right]_{21}\exp\left(n\left((1-\kappa)\mathfrak{g}(y) + (1-\kappa)\mathfrak{g}(x) + \delta V(x) + \ell_1\right)\right), \tag{2-85}$$

$$\left[\mathbf{Y}(y)^{-1}\mathbf{Y}(x)\right]_{31} = \left[\mathbf{W}(y)^{-1}\mathbf{W}(x)\right]_{31}\exp\left(n\left((1-\kappa)\mathfrak{g}(x) + \delta V(x) + \delta V(y) + l_2 - \eta\right)\right) \tag{2-86}$$

for $x$ and $y$ in $\Omega_1$. We have

$$\mathbf{W}(z) = (\mathbf{1} + \mathcal{O}(n^{-(1-\gamma)/2})\mathbf{\Psi}^\infty(z) = (\mathbf{1} + \mathcal{O}(n^{-(1-\gamma)/2})\mathbf{\Psi}(z)\mathbf{R}(\zeta(z)) \tag{2-87}$$

for $z \in \mathbb{D}_{a^\star}$. From (2-42), we have

$$\mathbf{\Psi}(y)^{-1}\mathbf{\Psi}(x) = \mathbf{I} + \mathcal{O}\left((\zeta_x - \zeta_y)\kappa^{1/2}\right). \tag{2-88}$$

We define the functions $\mathcal{Q}_i(z;\kappa)$ to be:

$$\begin{aligned}
\mathcal{Q}_1(z;\kappa) &:= -V(z) + 2(1-\kappa)\mathfrak{g}(z;\kappa) + \ell_1(\kappa), &(2\text{-}89)\\
\mathcal{Q}_2(z;\kappa) &:= -V(z) + az + (1-\kappa)\mathfrak{g}(z;\kappa) + l_2, &(2\text{-}90)\\
\mathcal{Q}_3(z;\kappa) &:= az - (1-\kappa)\mathfrak{g}(z;\kappa) - \ell_1(\kappa) + l_2. &(2\text{-}91)
\end{aligned}$$

Now combining (2-85), (2-87), (2-88), (2-63), $\det \mathbf{R}(\zeta) = 1$, and noting the $\mathcal{O}(\kappa)$ error terms from (2-88) are subsumed by the $\mathcal{O}(n^{-(1-\gamma)/2})$ error terms from (2-87) gives

$$e^{-\frac{n}{2}(V(x)+V(y))}\left[\mathbf{Y}(y)^{-1}\mathbf{Y}(x)\right]_{21} = \left(\mathcal{O}\left((\zeta_x-\zeta_y)\kappa^{1/2}\right)\cdot H_r^{(r)}(\zeta_x - \zeta_0) + \mathcal{O}(n^{-(1-\gamma)/2})\right)e^{n(*)}, \tag{2-92}$$

where

$$\begin{aligned}
(*) =& \frac{1}{2}\mathcal{Q}_1(x) + \frac{1}{2}\mathcal{Q}_1(y) + \delta V(x) - \kappa\log\zeta_x - \kappa\sum_{j=1}^{K}\frac{c_j^{(H)}}{\zeta_x^j}\\
=& \frac{1}{2}P_1(x) + \frac{1}{2}P_1(y) + \mathcal{O}(\kappa\ln\kappa).
\end{aligned} \tag{2-93}$$

The last equality is shown by noticing that rearranging the terms in (2-43) we have

$$\delta V(x) - \kappa\log\zeta_x - \kappa\sum_{j=1}^{K}\frac{c_j^{(H)}}{\zeta_x^j} = \frac{1}{2}\left(V(x) - ax - (1-\kappa)\mathfrak{g}(x) - \ell_2 - \frac{\kappa}{2}(\zeta-\zeta_0)^2 + \kappa\ln\kappa + \mathfrak{b}\right) \tag{2-94}$$

Since the real part of $P_1(z)$ is negative for $z$ near $a^\star$ (Lemma 2.1(b)), for $\kappa$ sufficiently small the real part of the exponent in (2-92) is negative.

Define

$$F_r^{\mathrm{GUE}}(\zeta_x, \zeta_y) := \frac{2\pi i}{k_{r-1}^{(r)}} \left( H_r^{(r)}(\zeta_x - \zeta_0) H_{r-1}^{(r)}(\zeta_y - \zeta_0) - H_{r-1}^{(r)}(\zeta_x - \zeta_0) H_r^{(r)}(\zeta_y - \zeta_0) \right). \tag{2-95}$$

From (2-86),

$$e^{-\frac{n}{2}(V(x)+V(y))+nay} \left[\mathbf{Y}(y)^{-1}\mathbf{Y}(x)\right]_{31} = \left( F_r^{\mathrm{GUE}}(\zeta_x, \zeta_y) + \mathcal{O}(n^{-(1-\gamma)/2}) \right) e^{n(**)}, \tag{2-96}$$

where from (2-86)

$$\begin{aligned}(**) = -\frac{V(x)}{2} - \frac{V(y)}{2} + ay + \delta V(y) + \delta V(x) - \kappa \ln \zeta_x - \kappa \sum_{j=1}^{K} \frac{c_j^{(H)}}{\zeta_x^j} - \kappa \ln \zeta_y - \kappa \sum_{j=1}^{K} \frac{c_j^{(H)}}{\zeta_y^j} + \\ + (1-\kappa)\mathfrak{g}(x) + \kappa \ell_H \,.\end{aligned} \tag{2-97}$$

Using now (2-94) (for both $x$ and $y$) and rearranging the terms we find

$$\begin{aligned}(**) = \quad & \frac{1}{2}\left[ay - (1-\kappa)\mathfrak{g}(y) - \frac{\kappa}{2}(\zeta_y - \zeta_0)^2 - l_2 + \kappa \ln \kappa + \mathfrak{b}\right] + \\ & + \frac{1}{2}\left[(1-\kappa)\mathfrak{g}(x) - ax - \frac{\kappa}{2}(\zeta_x - \zeta_0)^2 - l_2 + \kappa \ln \kappa + \mathfrak{b}\right] + \kappa \ell_H - \eta \qquad (2\text{-}98) \\ = \quad & -\frac{\kappa}{4}(\zeta_x - \zeta_0)^2 - \frac{\kappa}{4}(\zeta_y - \zeta_0)^2 - \frac{1}{2}\mathcal{Q}_3(x) + \frac{1}{2}\mathcal{Q}_3(y). \qquad (2\text{-}99)\end{aligned}$$

where we have used the definition of $\eta := \kappa \ln \kappa + \mathfrak{b} + \kappa \ell_H$ (2-48). From this and the fact that (2-92) is exponentially decaying in $n$ shows

$$K_n(x,y) = \frac{e^{-\frac{n}{2}\mathcal{Q}_3(x) + \frac{n}{2}\mathcal{Q}_3(y)}}{2\pi i(x-y)} \left( F_r^{\mathrm{GUE}}(\zeta_x, \zeta_y) + \mathcal{O}(n^{-(1-\gamma)/2}) \right) e^{-\frac{r}{4}(\zeta_x - \zeta_0)^2 - \frac{r}{4}(\zeta_y - \zeta_0)^2}. \tag{2-100}$$

One final application of (2-42) (to switch $x - y$ to $\zeta_x - \zeta_y$) and Proposition 2.1 (to show convergence of $\mathcal{Q}_3$ to $P_3$) gives (1-20).

□

# 3 The subcritical regime

We now take $V(x)$ and $a$ so Definition 1.2 of the subcritical regime is satisfied; that is $a < a_c$ and the function $\Re P_2$ has no global maximum on $\mathbb{R} \setminus [\alpha, \beta]$. In this case $\Re P_3$ has a (unique) global minimum at $z = b^\star$ (with value zero, as per our choice of $l_2$ in Definition 1.5). We will show that almost surely there are no outliers. We will freely reuse the same notation from Section 2.1 for new objects which played a similar role in the analysis of the supercritical regime. To begin, fix $\gamma \in [0, 1)$ and again set $K$ to be the smallest nonnegative integer satisfying

$$K \geq \max\left\{\frac{3\gamma - 1}{1 - \gamma}, 0\right\} \tag{3-1}$$

## 3.1 Modified equilibrium problem (subcritical case)

The procedure here parallels closely the one followed in the supercritical case, and hence we will only state the results since their proof does not differ significantly from the other case.

Let $J$ be a closed subset not containing the point $b^\star$ and containing $[\alpha, \beta]$ in its interior; we recall that $b^\star(a) > \beta$ for $0 < a < a_c$.

**Proposition 3.1.** *For any $K \in \mathbb{N}$ there is a neighborhood of the origin in $(\kappa, \vec{\delta}) \in \mathbb{C}^{1+K}$ such that the equilibrium measure $\widetilde{\sigma}(x)\mathrm{d}x$ of* **unit** *total mass for the external field*

$$\widetilde{V}(z) := V(z) + \delta V(z), \quad \delta V(z) := \kappa \ln(z - b^\star) + \kappa \sum_{j=1}^{K} \frac{\delta_j}{2(z - b^\star)^j} \tag{3-2}$$

*is supported on a single interval $[\alpha(\kappa, \vec{\delta}), \beta(\kappa, \vec{\delta})]$ still contained in the interior of $J$: the endpoints $\alpha(\kappa, \vec{\delta}), \beta(\kappa, \vec{\delta})$ are analytic functions of the specified variables. Furthermore the $g$–function of this problem*

$$\mathfrak{g}(z) := \int \ln(z - w)\widetilde{\sigma}(w)\mathrm{d}w \tag{3-3}$$

*converges uniformly over closed subsets not containing $[\alpha, \beta]$ to the unperturbed $g$–function.*

The proof is identical to that of Proposition 2.1: the only difference is that now the modified equilibrium measure is of unit total mass, rather than of mass $1 - \kappa$. We next re-define the three functions $\mathcal{P}_j$'s; the definition is subtly different from the previous (2-12), (2-13),(2-14) and hence there is a possibility of confusion for the reader. The advantage is that we will be able to recycle many of the previous computations.

$$\mathcal{P}_1(z) := -\widetilde{V}(z) + 2\mathfrak{g}(z) + \ell_1, \tag{3-4}$$

$$\mathcal{P}_2(z) := \mathcal{P}_1(z) + \mathcal{P}_3(z) = -V(z) + \delta V(z) + az + \mathfrak{g}(z) + l_2 \tag{3-5}$$

$$\mathcal{P}_3(z) := az - \mathfrak{g}(z) + 2\delta V(z) + l_2 - \ell_1 = \mathcal{P}_2(z) - \mathcal{P}_1(z) \tag{3-6}$$

$$\widetilde{V}(z) := V(z) + \delta V(z)\ , \qquad \delta V(z) := \kappa \log(z - b^\star) + \sum_{j=1}^{2k} \frac{\delta_j}{2(z - b^\star)^j} + \ell_1 \tag{3-7}$$

It is **important** to point out the change of sign in the definition of $\widetilde{V}$, relative to the supercritical case. We also remind that $\ell_1, \mathfrak{g}$ are analytic functions of $\kappa, \vec{\delta}$, while $l_2$ is the constant mandated in Definition 1.5.

**Theorem 3.1.** *There exists a conformal change of coordinate $\rho = \rho(z; \kappa, \vec{\delta})$* fixing $z = b^\star$ *($\rho(b^\star; \kappa, \vec{\delta}) \equiv 0$) that depends analytically on the parameters $\kappa, \vec{\delta}$ such that*

$$\mathcal{P}_3(z) := az - \mathfrak{g}(z; \kappa, \vec{\delta}) + l_2 - \ell_1 + 2\kappa \ln(z - b^\star) + \sum_{j=1}^{K} \frac{\delta_j}{(z - b^\star)^j} \tag{3-8}$$

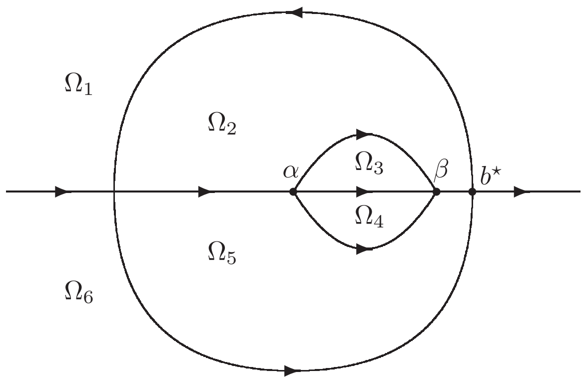


Figure 3: The regions $\Omega_i$ and the oriented contour $\Gamma$ for the subcritical case.

*can be written as*

$$\mathcal{P}_3(z;\kappa,\vec{\delta}) = \frac{1}{2}(\rho-\mathfrak{a})^2 + 2\kappa\ln\rho + \mathfrak{b} + \sum_{j=2}^{K}\frac{\gamma_j}{\rho^j} \tag{3-9}$$

*where the parameters* $\mathfrak{a} = \mathfrak{a}(\kappa,\vec{\delta})$, $\mathfrak{b} = \mathfrak{b}(\kappa,\vec{\delta})$ *and* $\vec{\gamma} = \vec{\gamma}(\kappa;\vec{\delta})$ *are analytic functions of the indicated parameters. Furthermore the Jacobian*

$$\frac{\partial\vec{\gamma}}{\partial\vec{\delta}} \tag{3-10}$$

*is nonsingular in a neighborhood of the origin (for* $\kappa$ *sufficiently small).*

**Theorem 3.2.** *There exists a conformal change of coordinate* $\zeta(z;\kappa)$ *of the form*

$$\zeta(z;\kappa) = \frac{\rho(z;\kappa)}{i\sqrt{\kappa}} = \frac{1}{i\sqrt{\kappa}}C(z-b^\star)(1+\mathcal{O}(z-b^\star))\ ,C>0 \tag{3-11}$$

*and a choice of* $\vec{\delta} = \vec{\delta}(\kappa)$ *for the deformed potential (2-2) in Puiseux series of* $\sqrt{\kappa}$ *such that*

$$\mathcal{P}_3(z;\kappa,\vec{\delta}(\kappa)) = -\frac{\kappa}{2}(\zeta-\zeta_0)^2 + 2\kappa\ln(\sqrt{\kappa}\zeta) + \mathfrak{b} + 2\kappa\sum_{j=1}^{K}\frac{c_j^{(H)}}{\zeta^j}\ . \tag{3-12}$$

*The functions* $\zeta_0(\kappa), \beta(\kappa), \vec{\delta}(\kappa)$ *admit a Puiseux expansion and are of orders*

$$\zeta_0 = \mathcal{O}(\sqrt{\kappa})\ ,\quad \beta = \mathcal{O}(\kappa)\ ,\quad \vec{\delta} = \mathcal{O}(\kappa). \tag{3-13}$$

*The expressions* $c_j^{(H)}$ *are polynomials of degree* $j$ *in* $\zeta_0$ *determined by the formula* (2-36).

## 3.2 Steepest descent analysis (subcritical case)

The regions $\Omega_i$, $i = 1,...6$ are defined in Figure 3. Following the example of the supercritical case, we introduce the **same** new matrix $\mathbf{W}$ as in (2-45). Note, however that the definition of the regions $\Omega_j$'s now follows Figure 3, the constant $l_2$ follows Definition 1.5 in the subcritical case and $\delta V$ is given in (3-2) instead.

The new matrix $\mathbf{W}(z;\kappa)$ satisfies the same jump conditions (2-49) as in the supercritical case (with, of course, the new definitions of the $\mathcal{P}_j(z;\kappa)$'s and $\Omega_j$'s) as well as the same asymptotic condition (2-50). Due to the new definitions of $\mathcal{P}_j$'s (3-4), (3-5), (3-6) , the behavior near $z = b^\star$ is now different:

$$\mathbf{W} = (\text{analytic}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{-n\delta V(z)} & 0 \\ 0 & 0 & \mathrm{e}^{n\delta V(z)} \end{pmatrix} \quad \text{as } z \to b^\star. \tag{3-14}$$

The outer parametrix problem is the same as in the supercritical case, so we again define the outer parametrix solution $\mathbf{\Psi}(z;\kappa)$ as in (2-53).

### 3.2.1 The local parametrix near $b^\star$

Define $\mathbb{D}_{b^\star}$ to be a fixed-size circular disk centered at $b^\star$ which is small enough so that it does not intersect the inner lenses, and

$$\Re(P_2) < 0 \quad \text{for} \quad z \in \mathbb{D}_{b^\star}. \tag{3-15}$$

This last condition is possible for $\kappa$ sufficiently small because –due to the definition of $l_2$ in Definition 1.5 for the subcritical case– $P_2(b^\star) < 0$. We now use Theorem 3.2; in a local coordinate centered at $b^\star$, the analytic part of $\mathcal{P}_3(z;\kappa)$ behaves the same way (quadratically with a maximum at the origin) along the imaginary axis as $\mathcal{P}_2(z;\kappa)$ did along the real axis in the supercritical regime. The scaling of $\zeta$ is analogous to (2-59);

$$\zeta = \mathcal{O}\left(n^{(1-\gamma)/2}\right) \quad \text{when } z \in \partial\mathbb{D}_{b^\star}. \tag{3-16}$$

$$\mathcal{P}_3 = f(z;\kappa) + 2\delta V(z) + l_2 - \ell_1 = -\frac{\kappa}{2}(\zeta-\zeta_0)^2 + 2\kappa\log\zeta + 2\kappa\sum_{j=1}^{k}\frac{c_j^{(H)}}{\zeta^{2j}} + \kappa\ln\kappa + \mathfrak{b}. \tag{3-17}$$

**Definition 3.1.** *The local parametrix within the disk $\mathbb{D}_{b^\star}$ shall be the unique solution $\mathbf{R}(z)$ to the following model Riemann–Hilbert problem:*

$$\begin{cases} \mathbf{R}_+(\zeta) = \mathbf{R}_-(\zeta)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\zeta^{2r}\exp\left(-\frac{r}{2}(\zeta-\zeta_0)^2 + r\ell_H + 2r\sum_{j=1}^{k}\frac{c_j^{(H)}}{\zeta^{2j}}\right) \\ 0 & 0 & 1 \end{pmatrix} \\ \qquad = \mathbf{R}_-(\zeta)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -e^{n\mathcal{P}_3} \\ 0 & 0 & 1 \end{pmatrix}, \quad \zeta \in \mathbb{R}, \\ \mathbf{R}(\zeta) = \mathbf{I} + \mathcal{O}\left(\frac{1}{\zeta}\right) \quad \textit{as } \zeta \to \infty, \\ \mathbf{R}(\zeta) = (\textit{analytic})\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta^{-r}\exp\left(-r\sum_{j=1}^{k}\frac{c_j^{(H)}}{\zeta^{2j}}\right) & 0 \\ 0 & 0 & \zeta^{r}\exp\left(r\sum_{j=1}^{k}\frac{c_j^{(H)}}{\zeta^{2j}}\right) \end{pmatrix} \quad \textit{as } \zeta \to 0. \end{cases} \tag{3-18}$$

This problem is almost the same as (2-60). Analogously to (2-63), the solution is ($\ell_H = -1 - 2\ln 2$ as in (2-34))

$$\mathbf{R}(\zeta) = \exp\left(-\frac{r}{2}\ell_H \mathbf{\Lambda}_{23}\right) \mathbf{H}_{23}(\zeta)\, \zeta^{-r\mathbf{\Lambda}_{23}}\ \exp\left(\left(\frac{r}{2}\ell_H - r\sum_{j=1}^{k} \frac{c_j^{(H)}}{\zeta^{2j}}\right)\mathbf{\Lambda}_{23}\right), \tag{3-19}$$

where

$$\mathbf{H}_{23}(\zeta) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & H_r^{(r)}(\zeta-\zeta_0) & \dfrac{-1}{2\pi i}\displaystyle\int_{-\infty}^{\infty} \frac{H_r^{(r)}(s-\zeta_0)e^{-\frac{r}{2}s^2}}{s-\zeta}ds \\ 0 & \dfrac{2\pi i}{-k_{r-1}^{(r)}} H_{r-1}^{(r)}(\zeta-\zeta_0) & \dfrac{1}{k_{r-1}^{(r)}}\displaystyle\int_{-\infty}^{\infty} \frac{H_{r-1}^{(r)}(s-\zeta_0)e^{-\frac{r}{2}s^2}}{s-\zeta}ds \end{pmatrix} \quad \text{and} \quad \mathbf{\Lambda}_{23} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{3-20}$$

Again the polynomials $H_m^{(r)}(\zeta)$ and the normalization constants $k_m^{(r)}$ are defined by (2-62). The analysis in the supercritical regime leading to (2-68) applies here as well, leading to

$$\mathbf{R}(\zeta) = \mathbf{I} + \mathcal{O}\left(\frac{1}{\zeta}\right) + \mathcal{O}\left(\frac{r}{\zeta^{2k+2}}\right). \tag{3-21}$$

## 3.3 The subcritical error analysis

Let $\mathbb{D}_\alpha$ and $\mathbb{D}_\beta$ be small, closed disks of fixed radii centered at $\alpha$ and $\beta$ that are bounded away from the outer lenses and $\mathbb{D}_{b^\star}$. Orient the boundaries $\partial\mathbb{D}_\alpha$ and $\partial\mathbb{D}_\beta$ clockwise. Let $\mathbf{P}_{\text{Ai}}^{(\alpha)}$ and $\mathbf{P}_{\text{Ai}}^{(\beta)}$ be the Airy parametrices constructed in $\mathbb{D}_\alpha$ and $\mathbb{D}_\beta$, respectively (see Section 2.4). Define the global parametrix $\mathbf{\Psi}^\infty(z)$ by

$$\mathbf{\Psi}^\infty(z) := \begin{cases} \mathbf{\Psi}(z), & z \notin \mathbb{D}_\alpha \cup \mathbb{D}_\beta \cup \mathbb{D}_{b^\star}, \\ \mathbf{\Psi}(z)\mathbf{R}(\zeta(z)), & z \in \mathbb{D}_{b^\star}, \\ \mathbf{P}_{\text{Ai}}^{(\alpha)}(z), & z \in \mathbb{D}_\alpha, \\ \mathbf{P}_{\text{Ai}}^{(\beta)}(z), & z \in \mathbb{D}_\beta. \end{cases} \tag{3-22}$$

The error matrix $\mathbf{E}(z)$ is given by

$$\mathbf{E}(z) := \mathbf{W}(z)\mathbf{\Psi}^\infty(z)^{-1}. \tag{3-23}$$

Let $\Gamma$ denote the contours given by the boundaries of the regions $\Omega_j$ in Figure 3. The error matrix satisfies a Riemann-Hilbert problem with jump matrix $\mathbf{V}^{(\mathbf{E})}(z)$ on the contours $\Gamma^{(\mathbf{E})}$ shown in Figure 4. The form of the jump matrix is as follows:

- For $z$ outside the disks $\mathbb{D}_\alpha$, $\mathbb{D}_\beta$, and $\mathbb{D}_{b^\star}$, and excluding the band $[\alpha,\beta]$:

$$\mathbf{V}^{(\mathbf{E})}(z) = \mathbf{\Psi}(z)\mathbf{V}^{(\mathbf{W})}(z)\mathbf{\Psi}(z)^{-1}, \quad z \in \Gamma \cap (\mathbb{D}_\alpha \cup \mathbb{D}_\beta \cup \mathbb{D}_{b^\star})^c \cap [\alpha,\beta]^c, \tag{3-24}$$

where $\mathbf{V}^{(\mathbf{W})}(z)$ is given by the formulas in (2-49).

- For $z$ on the boundaries of the disks $\partial\mathbb{D}_\alpha$, $\partial\mathbb{D}_\beta$, and $\partial\mathbb{D}_{b^\star}$:

$$\mathbf{V}^{(\mathbf{E})}(z) = \begin{cases} \mathbf{\Psi}(z)\mathbf{R}(\zeta)\mathbf{\Psi}(z)^{-1}, & z \in \partial\mathbb{D}_{b^\star}, \\ \mathbf{P}_{\text{Ai}}^{(\alpha)}(z)\mathbf{\Psi}(z)^{-1}, & z \in \partial\mathbb{D}_\alpha, \\ \mathbf{P}_{\text{Ai}}^{(\beta)}(z)\mathbf{\Psi}(z)^{-1}, & z \in \partial\mathbb{D}_\beta. \end{cases} \tag{3-25}$$

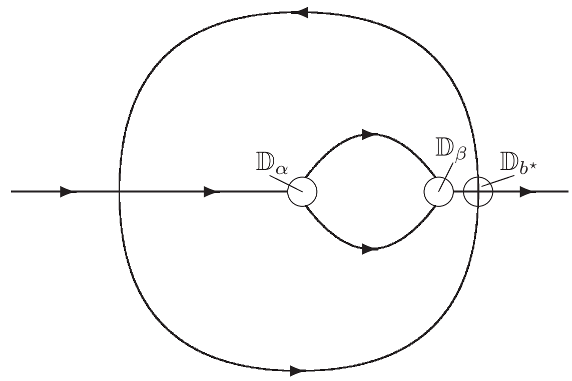


Figure 4: The jump contours $\Gamma^{(\mathbf{E})}$ for the Riemann-Hilbert problem for $\mathbf{E}(z)$ in the subcritical case.

- For $z$ inside the disk $\mathbb{D}_{b^\star}$:

$$\mathbf{V}^{(\mathbf{E})}(z) = \begin{cases} \mathbf{\Psi}(z)\mathbf{R}(\zeta)\begin{pmatrix} 1 & e^{n\mathcal{P}_1(z)} & e^{n\mathcal{P}_2(z)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\mathbf{R}(\zeta)^{-1}\mathbf{\Psi}(z)^{-1}, & z \in (\partial\Omega_1 \cap \partial\Omega_6) \cap \mathbb{D}_{b^\star}, \\ \mathbf{\Psi}(z)\mathbf{R}(\zeta)\begin{pmatrix} 1 & e^{n\mathcal{P}_1(z)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\mathbf{R}(\zeta)^{-1}\mathbf{\Psi}(z)^{-1}, & z \in (\partial\Omega_2 \cap \partial\Omega_5) \cap \mathbb{D}_{b^\star}. \end{cases} \tag{3-26}$$

- Furthermore, $\mathbf{V}^{(\mathbf{E})}(z) = \mathbf{I}$ on the contours

$$[\alpha,\beta] \cap (\mathbb{D}_\alpha \cup \mathbb{D}_\beta)^c, \quad \Gamma \cap \mathbb{D}_\alpha, \quad \Gamma \cap \mathbb{D}_\beta, \quad (\partial\Omega_1 \cap \partial\Omega_2) \cap \mathbb{D}_{b^\star}, \quad \text{and} \quad (\partial\Omega_5 \cap \partial\Omega_6) \cap \mathbb{D}_{b^\star}.$$

We now show that all of the jump matrices in (3-24)–(3-26) are uniformly close to the identity as $n \to \infty$.

Here are the results we will need for $P_1$, $P_2$, and $P_3$, analogous to Lemma 2.1:

**Lemma 3.1.** *In the subcritical regime, the inner and outer lenses can be chosen such that*

*(a) On the inner lenses outside of the disks around $\alpha$ and $\beta$: The real part of $P_1(z)$ is positive and bounded away from zero for $z \in [(\partial\Omega_2 \cap \partial\Omega_3) \cup (\partial\Omega_4 \cap \partial\Omega_5)] \cap (\mathbb{D}_\alpha \cup \mathbb{D}_\beta)^c$.*

*(b) On the real axis outside of $[\alpha,\beta]$ and the disks around $\alpha$ and $\beta$: The real part of $P_1(z)$ is negative and bounded away from zero for $z \in [(\partial\Omega_1 \cap \partial\Omega_6) \cup (\partial\Omega_2 \cap \partial\Omega_5)] \cap (\mathbb{D}_\alpha \cup \mathbb{D}_\beta)^c$.*

*(c) On the outer lenses outside of the disk around $b^\star$: For $\kappa$ sufficiently small, the real part of $P_3(z)$ is negative and bounded away from zero for $z \in [(\partial\Omega_1 \cap \partial\Omega_2) \cup (\partial\Omega_5 \cap \partial\Omega_6)] \cap \mathbb{D}^c_{b^\star}$.*

*(d) On the real axis outside of the outer lenses or on the real axis inside $\mathbb{D}_{b^\star}$: For $\kappa$ sufficiently small, the real part of $P_2(z)$ is negative and bounded away from zero for $z \in (\partial\Omega_1 \cap \partial\Omega_6) \cup [(\partial\Omega_2 \cap \partial\Omega_5) \cap \mathbb{D}_{b^\star}]$.*

*Proof.* Parts (a) and (b) follow from the analysis of the Riemann-Hilbert problem for the standard (non-multiple) orthogonal polynomials (for example, [21]).

For (c), first note $P_3(b^\star)=0$. Following the proof of Lemma 2.1(d), the outer lenses (defined in this regime to be a circle centered below $\alpha$ and passing through $b^\star$, that is big enough such that $\Re P_2$ is negative on the real axis to the left of the circle) are descent lines of $\operatorname{Re} P_3(z)$ for $\kappa$ sufficiently small. The result follows.

For (d), start with $\kappa=0$. Consider real $z$ to the left of the outer lenses. From Lemma 3.1(c), $\operatorname{Re} P_3<0$ at the left-most point of the outer lenses. Thus $\operatorname{Re} P_3(z)<0$ for such $z$ since $\operatorname{Re} P_3$ is a strictly increasing function for $z\in(-\infty,\alpha)$. Since $\operatorname{Re} P_1(z)$ is also negative here by construction, this means $\operatorname{Re} P_2(z)=\operatorname{Re}(P_1(z)+P_3(z))$ is also negative. This is also true for real $z$ inside $\mathbb{D}_{b^\star}$ by (3-15). Next, consider real $z$ to the right of the outer lenses. By Definition 1.2 we have $\operatorname{Re} P_2(z)<\operatorname{Re} P_3(b^\star)=0$. Along with the fact that $\operatorname{Re} P_1(z)<0$ here shows the desired result. □

Next come the necessary results for $\mathcal{P}_1(z;\kappa)$, $\mathcal{P}_2(z;\kappa)$, and $\mathcal{P}_3(z;\kappa)$ defined by (3-4)–(3-6). This lemma is analogous to Lemma 2.2 for the supercritical regime.

**Lemma 3.2.** *For $\kappa$ sufficiently small:*

*(a) On the two inner lenses outside of the disks around $\alpha$ and $\beta$: The real part of $\mathcal{P}_1(z;\kappa)$ is positive and bounded away from zero for $z\in[(\partial\Omega_2\cap\partial\Omega_3)\cup(\partial\Omega_4\cap\partial\Omega_5)]\cap(\mathbb{D}_\alpha\cup\mathbb{D}_\beta)^c$.*

*(b) On the real axis outside of $[\alpha,\beta]$ and the disks around $\alpha$, $\beta$, and $b^\star$: The real part of $\mathcal{P}_1(z;\kappa)$ is negative and bounded away from zero for $z\in[(\partial\Omega_1\cap\partial\Omega_6)\cup(\partial\Omega_2\cap\partial\Omega_5)]\cap(\mathbb{D}_\alpha\cup\mathbb{D}_\beta\cup\mathbb{D}_{b^\star})^c$.*

*(c) On the outer lenses outside of $\mathbb{D}_{b^\star}$: The real part of $\mathcal{P}_3(z;\kappa)$ is negative and bounded away from zero for $z\in[(\partial\Omega_1\cap\partial\Omega_2)\cup(\partial\Omega_5\cap\partial\Omega_6)]\cap\mathbb{D}_{b^\star}^c$.*

*(d) On the real axis outside of the outer lenses and $\mathbb{D}_{b^\star}$: The real part of $\mathcal{P}_2(z;\kappa)$ is negative and bounded away from zero for $z\in(\partial\Omega_1\cap\partial\Omega_6)\cap\mathbb{D}_{b^\star}^c$.*

*(e) On the real axis inside $\mathbb{D}_{b^\star}$: The real parts of*

$$\mathcal{P}_1(z;\kappa)-\kappa g_H(\zeta)+\kappa\log\zeta+\kappa\sum_{j=1}^{k}\frac{c_j^{(H)}}{\zeta^{2j}}\quad\text{and}\quad\mathcal{P}_2(z;\kappa)+\kappa g_H(\zeta)-\kappa\log\zeta-\kappa\sum_{j=1}^{k}\frac{c_j^{(H)}}{\zeta^{2j}}$$

*are negative and bounded away from zero for $z\in[(\partial\Omega_1\cap\partial\Omega_6)\cup(\partial\Omega_2\cap\partial\Omega_5)]\cap\mathbb{D}_{b^\star}$.*

*Proof.* Parts (a) and (b) come from Proposition 3.1 along with the fact that $\log(z-b^\star)$ and $(z-b^\star)^{-j}$, $j=1,\dots,2k$, is bounded outside of $\mathbb{D}_{b^\star}$. Part (c) comes from combining Lemma 3.1(c) with Proposition 3.1 and the boundedness of $\log(z-b^\star)$ and $(z-b^\star)^{-j}$, $j=1,\dots,2k$.

Part (d) follows from Lemma 3.1(d), Proposition 3.1 , and the boundedness of $\log(z-b^\star)$ and $(z-b^\star)^{-j}$, $j=1,\dots 2k$.

Finally, part (e) comes from (b) and (d) of Lemma 3.1 along with (3-11) and Proposition 3.1. □

These results allow us to now bound the jumps $\mathbf{V}^{(\mathbf{E})}$ of the error problem. Divide $\Gamma^{(\mathbf{E})}$ into a compact component $\Gamma_C^{(\mathbf{E})}$ and a noncompact component $\Gamma_N^{(\mathbf{E})}$:

$$\begin{aligned}\Gamma_C^{(\mathbf{E})}:=&\partial\mathbb{D}_\alpha\cup\partial\mathbb{D}_\beta\cup\partial\mathbb{D}_{b^\star}\cup(\Gamma\cap\mathbb{D}_{b^\star}),\\ \Gamma_N^{(\mathbf{E})}:=&\Gamma^{(\mathbf{E})}\backslash\Gamma_C^{(\mathbf{E})}.\end{aligned}\tag{3-27}$$

**Lemma 3.3.** *In the subcritical regime, for large $n$:*

*(a) Outside the disks $\mathbb{D}_\alpha$, $\mathbb{D}_\beta$, and $\mathbb{D}_{b^\star}$: There is a constant $c>0$ such that*

$$\mathbf{V}^{(\mathbf{E})}(z;\kappa) = \mathbf{I} + \mathcal{O}(e^{-cn}), \quad z \in \Gamma_N^{(\mathbf{E})}.$$

*(b) On the boundary of $\mathbb{D}_{b^\star}$:*

$$\mathbf{V}^{(\mathbf{E})}(z;\kappa) = \mathbf{I} + \mathcal{O}\left(n^{-(1-\gamma)/2}\right) + \mathcal{O}\left(n^{-k-1+(k+2)\gamma}\right), \quad z \in \partial\mathbb{D}_{b^\star}.$$

*(c) On the boundaries of $\mathbb{D}_\alpha$ and $\mathbb{D}_\beta$:*

$$\mathbf{V}^{(\mathbf{E})}(z;\kappa) = \mathbf{I} + \mathcal{O}\left(\frac{1}{n}\right), \quad z \in \partial\mathbb{D}_\alpha \cup \partial\mathbb{D}_\beta.$$

*(d) Inside $\mathbb{D}_{b^\star}$: There is a constant $c>0$ such that*

$$\mathbf{V}^{(\mathbf{E})}(z;\kappa) = \mathbf{I} + \mathcal{O}\left(e^{-cn}\right), \quad z \in \Gamma \cap \mathbb{D}_{b^\star}.$$

*Proof.* Part (a) is the result of Lemma 3.2(a)–(d) and the boundedness of $\mathbf{\Psi}(z)$. Part (b) is from (3-16), (3-21), and the boundedness of $\mathbf{\Psi}(z)$. Part (c) is from the construction of the parametrices $\mathbf{P}^{(\alpha)}_{\mathbf{Ai}}(z)$ and $\mathbf{P}^{(\beta)}_{\mathbf{Ai}}(z)$ (see, for instance, [24]).

For part (d), consider the jumps (3-26). By (3-19) for $\mathbf{R}(\zeta)$,

$$\begin{aligned}\mathbf{V}^{(\mathbf{E})}(z;\kappa) =& \mathbf{\Psi}(z)e^{-\frac{r}{2}\ell_H\mathbf{\Lambda}_{23}}\mathbf{H}_{23-}(\zeta)e^{-r(g_H(\zeta)-\frac{\ell_H}{2})\mathbf{\Lambda}_{23}}\begin{pmatrix}1 & (*)_{12} & (*)_{13}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\times\\ &\times e^{r(g_H(\zeta)-\frac{\ell_H}{2})\mathbf{\Lambda}_{23}}\mathbf{H}_{23-}(\zeta)^{-1}e^{\frac{r}{2}\ell_H\mathbf{\Lambda}_{23}}\mathbf{\Psi}(z)^{-1}, \quad z \in (\partial\Omega_1\cap\partial\Omega_6)\cap\mathbb{D}_{b^\star},\end{aligned} \tag{3-28}$$

and

$$\begin{aligned}\mathbf{V}^{(\mathbf{E})}(z;\kappa) =& \mathbf{\Psi}(z)e^{-\frac{r}{2}\ell_H\mathbf{\Lambda}_{23}}\mathbf{H}_{23-}(\zeta)e^{-r(g_H(\zeta)-\frac{\ell_H}{2})\mathbf{\Lambda}_{23}}\begin{pmatrix}1 & (*)_{12} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\times\\ &\times e^{r(g_H(\zeta)-\frac{\ell_H}{2})\mathbf{\Lambda}_{23}}\mathbf{H}_{23-}(\zeta)^{-1}e^{\frac{r}{2}\ell_H\mathbf{\Lambda}_{23}}\mathbf{\Psi}(z)^{-1}, \quad z \in (\partial\Omega_2\cap\partial\Omega_5)\cap\mathbb{D}_{b^\star},\end{aligned} \tag{3-29}$$

wherein

$$\begin{aligned}(*)_{12} &= \exp\left(n\mathcal{P}_1(z;\kappa) - rg_H(\zeta) + r\log\zeta + r\sum_{j=1}^{k}\frac{c_j^{(H)}}{\zeta^{2j}}\right),\\ (*)_{13} &= \exp\left(n\mathcal{P}_2(z;\kappa) + rg_H(\zeta) - r\log\zeta - r\sum_{j=1}^{k}\frac{c_j^{(H)}}{\zeta^{2j}}\right).\end{aligned} \tag{3-30}$$

This along with Lemma 3.2(e) and the boundedness of $\mathbf{\Psi}(z)$ in $\mathbb{D}_{b^\star}$ gives the result (d). ☐

We can now show that $\mathbf{E}(z)$ is asymptotically close to the identity. The proof of the following lemma follows that of Lemma 2.4:

**Lemma 3.4.** *In the subcritical regime, for $n$ large,*

$$\mathbf{E}(z) = \mathbf{I} + \mathcal{O}\left(n^{-(1-\gamma)/2}\right)$$

*uniformly in $z$.*

### 3.4 The subcritical kernel and proof of Theorem 1.2

*Proof of Theorem 1.2.* Once again, recall the kernel (1-18):

$$K_n(x,y) = \frac{e^{-\frac{1}{2}n(V(x)+V(y))}}{2\pi i(x-y)} \left( \left[\mathbf{Y}(y)^{-1}\mathbf{Y}(x)\right]_{21} + e^{nay}\left[\mathbf{Y}(y)^{-1}\mathbf{Y}(x)\right]_{31} \right). \tag{3-31}$$

While the function $\mathbf{Y}(z)$ has a jump for $z \in \mathbb{D}_{b^\star}$, the first column of $\mathbf{Y}(z)$ does not (see the Riemann-Hilbert problem (1-17)); observing the Riemann Hilbert Problem for $\mathbf{Y}^{-1}$ we also note that the second and third *rows* of $\mathbf{Y}^{-1}$ are entire functions. Therefore we can pick $x$ and $y$ to be in a convenient region. We choose $x$ and $y$ to be in $\Omega_1$ as defined in Figure 3.

From the transformation (2-45) (which is the same in the subcritical case as noted at the beginning of Section 3.2) and using the new definitions (3-4), (3-5), (3-6), we see that

$$\left[\mathbf{Y}(y)^{-1}\mathbf{Y}(x)\right]_{21} = \left[\mathbf{W}(y)^{-1}\mathbf{W}(x)\right]_{21} \exp\left(n\left(\mathfrak{g}(y)+\mathfrak{g}(x) - \delta V(y) + \ell_1\right)\right), \tag{3-32}$$

$$\left[\mathbf{Y}(y)^{-1}\mathbf{Y}(x)\right]_{31} = \left[\mathbf{W}(y)^{-1}\mathbf{W}(x)\right]_{31} \exp\left(n\left(\mathfrak{g}(x) + \delta V(y) + l_2 - \eta\right)\right) \tag{3-33}$$

for $x$ and $y$ in $\Omega_1$. As in the supercritical case, we have

$$\mathbf{W}(z) = \left(\mathbf{I} + \mathcal{O}\left(n^{-(1-\gamma)/2}\right)\right)\mathbf{\Psi}(z)\mathbf{R}(\zeta(z)) \tag{3-34}$$

and

$$\mathbf{\Psi}(y)^{-1}\mathbf{\Psi}(x) = \mathbf{I} + \mathcal{O}\left((\zeta_x - \zeta_y)\kappa^{1/2}\right). \tag{3-35}$$

Define $\mathcal{Q}_i(z;\kappa)$ to be $\mathcal{P}_i(z;\kappa)$ without the logarithm or pole terms:

$$\begin{aligned}
\mathcal{Q}_1(z;\kappa) &:= -V(z) + 2\mathfrak{g}(z;\kappa) + \ell_1, &(3\text{-}36)\\
\mathcal{Q}_2(z;\kappa) &:= -V(z) + az + \mathfrak{g}(z;\kappa) + l_2, &(3\text{-}37)\\
\mathcal{Q}_3(z;\kappa) &:= az - \mathfrak{g}(z;\kappa) + l_2 - \ell_1. &(3\text{-}38)
\end{aligned}$$

Recall (3-20) that

$$\mathbf{R}(\zeta) = \mathcal{O}(1)\mathrm{e}^{-n\left(\kappa\sum_{j=1}^{K}\frac{c_j^{(H)}}{\zeta^j} + \kappa\ln\zeta - \frac{\kappa}{2}\ell_H\right)\mathbf{\Lambda}_{23}} \quad \text{as } \zeta \to 0. \tag{3-39}$$

Now combining (3-32), (3-35), (3-39) gives

$$e^{-\frac{n}{2}(V(x)+V(y))}\left[\mathbf{Y}(y)^{-1}\mathbf{Y}(x)\right]_{21} = \mathcal{O}((x-y)n^{-(1-\gamma)/2})e^{n(*)} \tag{3-40}$$

and

$$e^{-\frac{n}{2}(V(x)+V(y))+nay}\left[\mathbf{Y}(y)^{-1}\mathbf{Y}(x)\right]_{31} = \mathcal{O}((x-y)n^{-(1-\gamma)/2})e^{n(**)}. \tag{3-41}$$

Rearranging the terms from (3-8) and (3-12) we find

$$\delta V(z) - \kappa\ln\zeta - \kappa\sum_{j=1}^{K}\frac{c_j^{(H)}}{\zeta^j} = \frac{1}{2}\left(\mathfrak{g}(z) - az - \frac{\kappa}{2}(\zeta-\zeta_0)^2 + \kappa\ln\kappa + \mathfrak{b} - l_2 + \ell_1\right). \tag{3-42}$$

Using (3-42) we can rewrite

$$(\star) = \frac{1}{2}\mathcal{Q}_1(x) + \frac{1}{2}\mathcal{Q}_2(y) + \frac{\kappa}{4}(\zeta_y - \zeta_0)^2 - \kappa \ln \sqrt{\kappa} - \frac{\mathfrak{b}}{2} = \frac{1}{2}P_1(x) + \frac{1}{2}P_2(y) + \mathcal{O}(\kappa \ln \kappa) \tag{3-43}$$

and

$$(\star\star) = \frac{1}{2}\mathcal{Q}_1(x) + \frac{1}{2}\mathcal{Q}_2(y) - \frac{\kappa}{4}(\zeta_y - \zeta_0)^2 - \frac{\eta}{2} + \kappa \ell_H = \frac{1}{2}P_1(x) + \frac{1}{2}P_2(y) + \mathcal{O}(\kappa \ln \kappa). \tag{3-44}$$

Here we have used Proposition 3.1 to convert $\mathfrak{g}$ to $g$. Since $\Re P_1(b^\star) < 0$ and also $\Re P_2(b^\star) < 0$, the theorem follows. □

# A The detailed analysis of Theorem 2.1

In this appendix we prove the existence of the local change of variables used in the supercritical and subcritical cases. We also demonstrate how the change of variables can be computed explicitly termwise.

## A.1 Background material

We start recalling that the space $\mathcal{H}(\mathbb{D}(r))$ of holomorphic functions on an open connected domain $\mathbb{D}$ (a disk of radius $r$ for simplicity) is a Banach space with respect to the sup norm.

The theorem of existence and uniqueness for ODEs can be extended to any Banach space $\mathcal{M}$. A sufficient condition for the integrability is the Lipshitz property, namely that we are given a (time-dependent) vector field $\mathcal{V} : \mathcal{M} \times J \to T\mathcal{M}$ which is **jointly continuous** and **Lipshitz**. Let

$$\Omega_1 := \{\zeta : \mathbb{D}(r) \to \mathbb{C},\ \zeta(0) = 0, \|\zeta\|_\infty < \infty, \zeta \ \textbf{univalent}\} \tag{A-1}$$

and

$$\Omega := \{\zeta : \mathbb{D}(r) \to \mathbb{C},\ \zeta(0) = 0, \|\zeta\|_\infty < \infty\}. \tag{A-2}$$

**Lemma A.1.** *The evaluation map of the inverse $\rho^{-1}$ at a point is locally Lipshitz on $\Omega_1$. More precisely:*

$$\forall \zeta_0 \in \Omega_1\ \exists C, \rho, S > 0 \ s.t.\ \forall \xi \in \mathbb{C},\ |\xi| < \rho \ \ \forall \zeta_1, \zeta_2 \in B_S(\zeta_0) \subset \Omega_1 \tag{A-3}$$

$$|\zeta_1^{-1}(\xi) - \zeta_2^{-1}(\xi)| \leq C\|\zeta_1 - \zeta_2\|_\infty = C \sup_{z \in \mathbb{D}(r)} |\zeta_1(z) - \zeta_2(z)| \ . \tag{A-4}$$

*Proof.* Note that $\Omega_1$ is an **open** subset of the Banach vector space of bounded analytic functions on $\mathbb{D}(r)$ containing the identity map. Therefore the Banach ball of radius $S > 0$ centered at $\zeta_0 \in \Omega_1$ lies all within $\Omega_1$ for sufficiently small $S$.

First we note that the forward map is locally Lipshitz; that is, let $z_0 \in \mathbb{D}(r/2)$, then any of the functionals $\zeta^{(n)}(z_0)$ are Lipshitz

$$|\zeta_1^{(n)}(z_0) - \zeta_2^{(n)}(z_0)| = \left| \frac{n!}{2i\pi} \oint_{|z|=2/3r} \frac{(\zeta_1(z) - \zeta_2(z))\mathrm{d}z}{(z - z_0)^{n+1}} \right| \leq \frac{1}{2\pi \left(\frac{2}{3} - \frac{1}{2}\right)^{n+1} r^n} \|\zeta_1 - \zeta_2\| \ . \tag{A-5}$$

Let now $\zeta_0(z) \in \Omega_1$ be univalent and let $0 < 3\rho := \inf_{|z|=r/2} |\zeta_0(z)|$. Let $\xi$ be such that $|\xi| < \rho$.

Let $\zeta_1, \zeta_2$ be two maps in a ball around $\zeta_0$ of radius $\rho$ ($\|\zeta_j - \zeta_0\| < \rho$) and consider (all integrations are on $|z| = \frac{1}{2}r$)

$$|z_0 - \widetilde{z}_0| := \left|\zeta_1^{-1}(\xi) - \zeta_2^{-1}(\xi)\right| = \left|\frac{1}{2i\pi}\oint \frac{z\zeta_1'(z)\mathrm{d}z}{\zeta_1(z)-\xi} - \frac{1}{2i\pi}\oint \frac{z\zeta_2'(z)\mathrm{d}z}{\zeta_2(z)-\xi}\right| \tag{A-6}$$

$$\leq \frac{1}{2\pi}\oint \left|\frac{z\left[(\zeta_1'(z)(\zeta_2-\xi) - \zeta_2'(z)(\zeta_1-\xi)\right]\mathrm{d}z}{(\zeta_1-\xi)(\zeta_2-\xi)}\right| \tag{A-7}$$

$$= \frac{1}{2\pi}\oint \left|\frac{z\left[\xi(\zeta_2'-\zeta_1') + \zeta_1'(\zeta_2-\zeta_1) + (\zeta_1'-\zeta_2')\zeta_1\right]\mathrm{d}z}{(\zeta_1-\xi)(\zeta_2-\xi)}\right| . \tag{A-8}$$

Since the derivative evaluation on the circle $z = r/2$ is uniformly Lipshitz, the above can be easily estimated by

$$\frac{1}{2\pi}\oint \left|\frac{z\left[\xi(\zeta_2'-\zeta_1') + \zeta_1'(\zeta_2-\zeta_1) + (\zeta_1'-\zeta_2')\zeta_1\right]\mathrm{d}z}{(\zeta_1-\xi)(\zeta_2-\xi)}\right| \leq \frac{Cr}{\inf\limits_{|z|=r/2}|\zeta_1-\xi| \inf\limits_{|z|=r/2}|\zeta_2-\xi|}\|\zeta_1-\zeta_2\| \tag{A-9}$$

where $C = \sup\{|\xi|, \sup_{|z|=r/2}|\zeta_1'|, \sup_{|z|=r/2}|\zeta_1|\}$. Since $\zeta_j$ are less than $\rho$ away from $\zeta_0$ we have that the two infima in the denominator are at least $\rho$, since $|\xi| < \rho$ and

$$\inf_{|z|=r/2}|\zeta_j| \geq \inf_{|z|=r/2}|\zeta_0| - \rho = 3\rho - \rho = 2\rho. \tag{A-10}$$

□

## A.2 Complete proof of Theorem 2.1 for the case $K = 0$

*Proof.* In this case there are no $\vec{\delta}$'s and no $\vec{\gamma}$'s; it should become clear that the general proof presents only notational complications, but is amenable to the same logic and hence the details are omitted. We also omit explicit reference to the dependence of $f(z)$ on $\kappa$ for brevity. We want to have

$$-f(z) + 2\kappa \ln z = -\frac{1}{2}(\rho(z) - \mathfrak{a})^2 + \mathfrak{b} + 2\kappa \ln \rho(z) , \tag{A-11}$$

where the goal is now to show that $\mathfrak{b} = \mathfrak{b}(\kappa)$, $\mathfrak{a} = \mathfrak{a}(\kappa)$ and $\rho = \rho(z;\kappa)$ are all analytic functions of $\kappa$, with $\rho$ being univalent in a neighborhood of $z = 0$ and mapping the origin to the origin. Consider the differentiation of the above identity with respect to $\kappa$:

$$-\dot{f}(z) + 2\ln z = \left(\mathfrak{a} - \rho + \frac{\kappa}{\rho}\right)\dot{\rho} + (\rho - \mathfrak{a})\dot{\mathfrak{a}} + \dot{\beta} + 2\ln\rho . \tag{A-12}$$

Solve for $\dot{\rho}$ and we find

$$\dot{\rho}(z;\kappa) = \rho\frac{(\rho-\mathfrak{a})\dot{\mathfrak{a}} + \dot{\mathfrak{b}} + 2\ln\left(\frac{\rho}{z}\right) + \dot{f}(z)}{\rho^2 - \mathfrak{a}\rho - \kappa} . \tag{A-13}$$

We want to view this equation as defining a vector field on a suitable Banach space that we define presently. Let $\Omega_1$ be the Banach **manifold** of **univalent, analytic functions** $\rho : \mathbb{D}(r) \to \mathbb{C}$ which fix the origin $\rho(0) = 0$; this is a closed Banach submanifold of all univalent analytic functions because the evaluation map is continuous. One only has to verify that if $\rho_n$ is a sequence of univalent analytic functions on $\mathbb{D}(r)$ converging in the sup-norm, the limit exists and it is still univalent. Define now

$$\mathcal{M} := \Omega_1 \times \mathbb{C}^2 = \{\mathbf{p} = (\rho, \mathfrak{a}, \mathfrak{b}), \quad \zeta \in \Omega_1, \quad \mathfrak{a}, \mathfrak{b} \in \mathbb{C}\} \ . \tag{A-14}$$

Formula (A-13) defines a vector field on $\mathcal{M}$: we will first explain it in coarse terms and then refine the details.

The denominator to (A-13) has two roots $\rho_1(\mathfrak{a}, \kappa), \rho_2(\mathfrak{a}, \kappa)$; since $\dot\rho$ must be also analytic, we must impose that the numerator vanishes at the same points, and hence

$$\dot{\mathfrak{a}} = A := \frac{\det \begin{bmatrix} -\dot f(z_1) - 2\ln(\rho_1/z_1) & 1 \\ -\dot f(z_2) - 2\ln(\rho_2/z_2) & 1 \end{bmatrix}}{\det \begin{bmatrix} \rho_1 - \mathfrak{a} & 1 \\ \rho_2 - \mathfrak{a} & 1 \end{bmatrix}} \tag{A-15}$$

$$\dot{\mathfrak{b}} = B := \frac{\det \begin{bmatrix} (\rho_1 - \mathfrak{a}) & -\dot f(z_1) - 2\ln(\rho_1/z_1) \\ (\rho_2 - \mathfrak{a}) & -\dot f(z_2) - 2\ln(\rho_2/z_2) \end{bmatrix}}{\det \begin{bmatrix} \rho_1 - \mathfrak{a} & 1 \\ \rho_2 - \mathfrak{a} & 1 \end{bmatrix}} \tag{A-16}$$

$$\rho_{1,2} := \frac{\mathfrak{a} \pm \sqrt{\mathfrak{a}^2 + 4\kappa}}{2}. \tag{A-17}$$

Here $z_1, z_2$ are the counterimages of $\rho_1, \rho_2$, $z_j := \rho^{-1}(\rho_j)$. Note that the expressions have analytic continuations to the case $\rho_1 = \rho_2$: indeed they are symmetric functions of the roots and therefore they can be expressed in terms of analytic functions of $\mathfrak{a}, \kappa$ (which play the role of elementary symmetric polynomials in the roots). Therefore we consider the (time dependent) vector field $\mathcal{V}$ on the manifold $\mathcal{M}$

$$\mathcal{V}([(\rho, \mathfrak{a}, \mathfrak{b}, \kappa)]) = (\eta(z), A, B) := \left( \rho \frac{(\rho - \mathfrak{a})A + B + 2\ln\left(\frac{\rho}{z}\right) + \dot f(z)}{\rho^2 - \mathfrak{a}\rho - \kappa}, A, B \right). \tag{A-18}$$

It is to be pointed out that $\eta(z) = \eta([\rho, \mathfrak{a}, \mathfrak{b}, \kappa]; z)$ is a tangent vector to $\Omega_1$, namely $\eta([\rho, \mathfrak{a}, \mathfrak{b}, \kappa]; 0) \equiv 0$.

The initial condition for the vector field is

$$\mathbf{p}_0 = (\rho(z;0), \mathfrak{a}(0), \mathfrak{b}(0)) = \left(\sqrt{2f(z)}, 0, 0\right). \tag{A-19}$$

Therefore the proof shall follow if we show that the vector field $\mathcal{V}$ is integrable in some neighborhood of the initial point and for sufficiently small values of $\kappa$ and for this to hold it is sufficient to verify the Lipshitz property.

**Lipshitz property for $\mathcal{V}$.** To complete the proof it is sufficient to show that the vector field is locally Lipshitz in a Banach neighborhood of the initial condition. The initial $\rho(z)$ is univalent in a small disk –say $\mathbb{D}(2r_0)$– around $z = 0$ because $\rho'(0) \neq 0$.

By simple continuity arguments in the sup norm, there is a sup-neighborhood $\mathcal{U}$ of $\rho$ consisting of univalent functions on $\mathbb{D}(r_0)$.

We therefore shall restrict $\mathfrak{a}, \kappa$ in such a way that $|\rho_j| < r_0$; this guarantees that we can *define* the components of $\mathcal{V}$. It is also quite clear that the restriction $|\rho_j(\mathfrak{a}, \kappa)| < r_0$ contains a polydisk in $\mathfrak{a}, \kappa$ (here $\mathfrak{b}$ is unrestricted). For example we can require

$$|\mathfrak{a}| < \frac{r_0}{10} \ , \ |\kappa| < \frac{{r_0}^2}{100} \ . \tag{A-20}$$

Thus, the neighborhood of the initial condition $\{\mathbf{p}_0\} \times \{0\}$ that we will analyze is

$$\mathfrak{V} := \left\{ (\mathbf{p}, \kappa) = (\rho, \mathfrak{a}, \mathfrak{b}, \kappa) \in \mathcal{M} \times \mathbb{R} : \quad \rho \in \mathcal{U} \ , \quad |\mathfrak{a}| < \frac{r_0}{10} \ , \ |\kappa| < \frac{{r_0}^2}{100} \right\} \tag{A-21}$$

The goal is now to prove that $\mathcal{V}$ is Lipshitz on $\mathfrak{V}$ for any fixed $t$. The fact that $A, B$ in (A-15, A-16) are Lipshitz functions follows from Lemma A.1. As for the first component, we have recall that the product of two Lipshitz **bounded** functions is Lipshitz, as well as the ratio if the denominator is bounded away from zero. By construction $\eta(z)$ (A-18) is analytic and hence its sup-norm is achieved on the boundary of $\mathbb{D}(r)$ (by the maximum modulus theorem). By the restrictions we made on $|\mathfrak{a}|$ and $|\kappa|$, the denominator is bounded away from zero on $\partial\mathbb{D}(r)$ and uniformly so with respect to the choice of $\rho$ in the Banach neighborhood $\mathcal{U}$ of $\mathfrak{a}$. On the other hand, the numerator is clearly Lipshitz. $\square$

# References


[1] M. Adler, J. Delépine, and P. van Moerbeke, Dyson's nonintersecting Brownian motions with a few outliers. *Commun. Pure Appl. Math.* **62** (2009), 334–395.

[2] M. Adler, N. Orantin, and P. van Moerbeke, Universality of the Pearcey process. arXiv:0901.4520 [math-ph] (2009).

[3] A. Aptekarev, P. Bleher, and A. Kuijlaars, Large $n$ limit of Gaussian random matrices with external source, part II. *Comm. Math. Phys.* **259** (2005), 367–389.

[4] J. Baik, G. Ben Arous, and S. Péché, Phase transition of the largest eigenvalue for non-null complex sample covariance matrices. *Ann. Probab.* **33** (2005), 1643–1697.

[5] J. Baik, and D. Wang. (To be published; Private communication.)

[6] J. Baik, Painleve formulas of the limiting distributions for non-null complex sample covariance matrices. *Duke Math. J.* **133** (2006), 205–235.

[7] J. Baik, On the Christoffel-Darboux kernel for random Hermitian matrices with external source. *Comput. Methods Funct. Theory* **9** (2009), 455–471

[8] M. Bertola, R. Buckingham, S.Y. Lee, and V. Pierce, Spectra of random Hermitian matrices with a small-rank external source: The critical regimes. Preprint 2010.

[9] M. Bertola and S. Lee, First colonization of a spectral outpost in random matrix theory. *Constr. Approx.* **30** (2009), no. 2, 225–263.

[10] M. Bertola and S. Lee, First colonization of a hard-edge in random matrix theory. *Constr. Approx.* **31** (2010), no. 2, 231–257.

[11] M. Bertola, S. Lee, and M. Yo, Mesoscopic colonization of a spectral band. *J. Phys. A* **42** (2009) no. 41, 415204, 17.

[12] P. Bleher, S. Delvaux, and A. Kuijlaars, Random matrix model with external source and a constrained vector equilibrium problem. arXiv:1001.1238 [math-ph] (2010).

[13] P. Bleher and A. Kuijlaars, Random matrices with external source and multiple orthogonal polynomials. *Int. Mat. Res. Not. IMRN* **3** (2004), 109–129.

[14] P. Bleher and A. Kuijlaars, Large $n$ limit of Gaussian random matrices with external source, part I. *Comm. Math. Phys.* **252** (2004), 43–76.

[15] P. Bleher and A. Kuijlaars, Large $n$ limit of Gaussian random matrices with external source, part III: double scaling limit. *Comm. Math. Phys.* **270** (2007), 481–517.

[16] E. Brézin and S. Hikami, Correlations of nearby levels induced by a random potential. *Nuclear Phys. B* **479** (1996), 697–706.

[17] E. Brézin and S. Hikami, Spectral form factor in a random matrix theory. *Phys. Rev. E* **55** (1997), 4067–4083.

[18] E. Brézin and S. Hikami, Extension of level-spacing universality. *Phys. Rev. E* **56** (1997), 264–269.

[19] E. Brézin and S. Hikami, Level spacing of random matrices in an external source. *Phys. Rev. E* **58** (1998), 7176–7185.

[20] E. Daems and A. Kuijlaars, Multiple orthogonal polynomials of mixed type and non-intersecting Brownian motions. *J. Approx. Theory* **146** (2007), 92–114.

[21] P. Deift, *Orthogonal Polynomials and Random Matrices: a Riemann-Hilbert Approach*. American Mathematial Society, Providence, 1998.

[22] P. Deift, T. Kriecherbauer, K. McLaughlin, New results on the equilibrium measure for logarithmic potentials in the presence of an external field. *J. Approx. Theory* **95** (1998), 388–475.

[23] P. Deift, T. Kriecherbauer, K. McLaughlin, S. Venakides, and X. Zhou, Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory. *Comm. Pure Appl. Math.* **52** (1999), 1335–1425.

[24] P. Deift, T. Kriecherbauer, K. McLaughlin, S. Venakides, and X. Zhou, Strong asymptotics of orthogonal polynomials with respect to exponential weights. *Comm. Pure Appl. Math.* **52** (1999), 1491–1552.

[25] S. Delvaux, and A. Kuijlaars, A graph-based equilibrium problem for the limiting distribution of non-intersecting Brownian motions at low temperature. *Constr. Approx.* (to be published) arXiv:0907.2310 [math.CV] (2009).

[26] N. Ercolani and K. McLaughlin, Asymptotics of the partition function for random matrices via Riemann-Hilbert techniques and applications to graphical enumeration. *Int. Math. Res. Not.* **14** (2003), 755–820.

[27] A. Fokas, A. Its, and A. Kitaev, The isomonodromy approach to matrix models in 2D quantum gravity. *Comm. Math. Phys.* **147** (1992), 395–430.

[28] K. McLaughlin, Asymptotic analysis of random matrices with external source and a family of algebraic curves. *Nonlinearity* **20** (2007), 1547–1571.

[29] S. Péché, The largest eigenvalue of small rank perturbations of Hermitian random matrices. *Probab. Theory Related Fields* **134** (2006), 127–173.

[30] C. Tracy and H. Widom, Level-spacing distributions and the Airy kernel. *Comm. Math. Phys.* **159** (1994), 151–174.

[31] E. Wigner, On the statistical distribution of the widths and spacings of nuclear resonance levels. *Proc. Cambridge Philos. Soc.* **47** (1951), 790–798.

[32] P. Zinn-Justin, Random Hermitian matrices in an external field. *Nuclear Phys. B* **497** (1997), 725–732.

[33] P. Zinn-Justin, Universality of correlation functions of Hermitian random matrices in an external field. *Comm. Math. Phys.* **194** (1998), 631–650.